Hilbert Functions of Cohen-Macaulay local rings

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Abstract. This survey is based on a series of lectures given by the author at the School in Commutative Algebra and its Connections to Geometry, Universidade Federal de Pernambuco, Olinda (Brasil), August 2009, in honor of V.W. Vasconcelos.

Contents

Introduction

The notion of Hilbert function is a central tool in commutative algebra and in algebraic geometry and is becoming increasingly important in combinatorics and in computational algebra. The Hilbert function of the homogeneous coordinate ring of a projective variety V was classically called the postulation of V and it is a rich source of discrete invariants of V and of its embedding. The dimension, the degree and the arithmetic genus of V, can be computed from the Hilbert function or from the Hilbert polynomial of its coordinate ring.

In this survey we mainly deal with the Hilbert function of Cohen-Macaulay local rings and our aim is to introduce the reader to some aspects of this area of dynamic mathematical activity.

2010 *Mathematics Subject Classification.* 13A02, 13A30,13C14, 13C15, 13H15.

Key words and phrases. Hilbert function, associated graded ring, Cohen-Macaulay ring, Gorenstein local ring, superficial sequence, multiplicity, Hilbert coefficients.

The author is personally grateful to the organizers of the School for giving her the opportunity to teach the favorite topic. A particular thank you goes to W. Vasconcelos because his teaching and his example have influenced her mathematical life greatly.

The Hilbert function of a local ring $(A, \mathfrak{m}, k)$ is, by definition, the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(A)$. The standard graded k-algebra $gr_{\mathfrak{m}}(A)$ arises from a relevant geometric construction and it has been studied extensively. Namely, if A is the localization at the origin of the coordinate ring of an affine variety V passing through 0, then $gr_{\mathfrak{m}}(A)$ is the coordinate ring of the *tangent cone* of V, which is the cone composed of all lines that are limiting positions of secant lines to V in 0. The $Proj$ of this algebra can also be seen as the *exceptional set* of the *blowing-up* of V in 0.

Despite the fact that the Hilbert function of a Cohen-Macaulay graded standard algebra is well understood by means of Macaulay's Theorem, very little is known in the local case. One of the main problems is whether geometric and homological properties of the local ring A can be carried on the corresponding tangent cone $gr_{\mathfrak{m}}(A)$. For example if a given local domain has fairly good properties, such as normality or Cohen-Macaulayness, its depth provides in general no information on the depth of the associated graded ring. It could be interesting to remind that a still wide open problem is to characterize the Hilbert function of an affine curve in $\mathbf{A^3}$ whose defining ideal is a complete intersection, while the Hilbert function of any complete intersection of homogeneous forms is well known in terms of the degrees of the generators.

In this presentation basic facts will be introduced, the use of fruitful techniques will be stressed in order to present easier proofs of known facts and to get a taste of some open problems. Starting from classical results by S. Abhyankar, D. Northcott and J. Sally, we present results on the coefficients of the Hilbert polynomial and, in several cases, we discuss algebraic and geometric properties of the local ring in terms of this asymptotic information. We shall focus our attention on the one-dimensional case because it plays an important role in our approach, in particular we provide tricky proofs proving the non-decreasing of the Hilbert function for several classes of one-dimensional Cohen-Macaulay local rings. The survey ends with results and techniques concerning the Hilbert function of Artinian Gorenstein local k-algebras. This investigation is strongly motivated by the interest related to the study of the Punctual Hilbert scheme and of the rationality of the Poincaré series of Gorenstein local rings (see for example [**6, 7, 8, 9, 16, 17, 36**]).

For more details and complete proofs we will refer to the monograph [**47**] written jointly with G. Valla, and to a recent paper [**14**] with J. Elias. For further reading on the same topic we would also suggest a series of lectures on problems and results on Hilbert functions of graded algebras given by G. Valla in Bellaterra (Spain) (see [**56**]), a survey by A. Corso and C. Polini on the Hilbert coefficients with a view toward blowup algebras (see [**10**]) and a survey on the Hilbert Functions of filtered modules by the author, 4-th Joint Seminar Japan-Vietnam, Meij University (2009) (see [**40**]). We present here several examples, all performed with the help of CoCoA [**61**].

1. Basic facts

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Denote by $\mu(I)$ the minimal number of generators of an ideal I of A. The *Hilbert* *function* of A is, by definition, the numerical function $HF_A : \mathbf{N} \to \mathbf{N}$ such that

$$HF_A(n) := \dim_k \ \mathfrak{m}^n/\mathfrak{m}^{n+1} = \mu(\mathfrak{m}^n)$$

for every $n \in \mathbf{N}$. Hence HF_A is also the Hilbert function of the homogeneous k-standard algebra

$$gr_{\mathfrak{m}}(A) = \oplus_{n\geq 0} \ \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

which is the associated graded ring of $\mathfrak{m}$.

The generating function of HF_A is a power series $HS_A(z) = \sum_{n\geq 0} HF_A(n)z^n$. This series, by the Hilbert-Serre Theorem, is rational and it can be written as

$$HS_A(z) = \frac{h_A(z)}{(1-z)^d},$$

where $h_A(z)$ is a polynomial with integer coefficients such that $h_A(1) \neq 0$. Moreover, d is the Krull dimension of A, which is also the Krull dimension of $gr_{\mathfrak{m}}(A)$. The numerator $h_A(z)$ is also called the h-polynomial of A. For large n, $HF_A(n)$ agrees with a polynomial $HP_A(X)$ with rational coefficients and degree $d-1$. It is called the Hilbert polynomial of A. We will denote by $\lambda(M)$ the length of M as A-module. Classically, another polynomial has been introduced, the Hilbert-Samuel polynomial of A, that is $\lambda(A/\mathfrak{m}^{n+1})$ for $n \gg 0$. It is denoted by $P_A(X)$ and

$$P_A(X) = \sum_{i=0}^{d} (-1)^i e_i(\mathfrak{m}) \binom{X+d-i}{d-i} \tag{1.1}$$

where $\binom{X+j}{j} := \frac{(X+j)(X+j-1)\cdots(X+1)}{j!}$.

The integers $e_0(\mathfrak{m}), e_1(\mathfrak{m}), \dots, e_d(\mathfrak{m})$ are uniquely determined by $\mathfrak{m}$ and are known as the Hilbert coefficients of A. In particular $e_0(\mathfrak{m})$ is the multiplicity of A. If there is not risk of confusion, we shall simply write e_i instead of $e_i(\mathfrak{m})$. We can prove that for every $i \geq 0$

$$e_i = \frac{h_A^{(i)}(1)}{i!}$$

where $0! = 1$ and $h_A^{(0)}(1) = h_A(1)$.

If $(\widehat{A}, \widehat{\mathfrak{m}})$ denotes the completion of A with respect to the m-adic filtration, it is well-known that

$$gr_{\widehat{\mathfrak{m}}}(\widehat{A}) \simeq gr_{\mathfrak{m}}(A)$$

as graded rings, hence studying the Hilbert function we may assume that A is complete. If A is equicharacteristic (for example if A is a k-algebra), we also assume $A = R/I$ where I is an ideal in the power series $R = k[[x_1, \dots, x_n]]$. This assumption is not restrictive by the well-known Cohen's theorem.

If $a \in A$ is a non zero element and $a \in \mathfrak{m}^r$, $a \notin \mathfrak{m}^{r+1}$, then we denote by $a^* := \overline{a} \in \mathfrak{m}^r/\mathfrak{m}^{r+1}$ and call it the *initial form* of a in $\mathrm{gr}_{\mathfrak{m}}(A)$. If $(R, \mathfrak{n})$ is a local ring and $A = R/I$, it is easy to prove that

$$gr_{\mathfrak{m}}(A) \simeq gr_{\mathfrak{n}}(R)/I^*$$

where I^* is the ideal generated by the initial forms of elements of I in $gr_{\mathfrak{n}}(R)$. If $R = k[[x_1, \dots, x_n]]$, then $\mathfrak{n}^r/\mathfrak{n}^{r+1} \simeq k[x_1, \dots, x_n]_r$, the k-vector space generated by all the forms of degree r, hence $gr_{\mathfrak{n}}(R) \simeq k[x_1, \dots, x_n]$.

If A is a Cohen-Macaulay graded k-algebra, the Hilbert Function is well-understood: its Hilbert series is

$$HS_A(z) = \frac{h_A(z)}{(1-z)^d}$$

where, by Macaulay's theorem, $h_A(z)$ is the Hilbert series of an Artinian graded k-algebra. The same does not hold true if A is a Cohen-Macaulay local ring.

The following example gives a measure of the complexity of the problem.

Example 1.1. Consider the coordinate ring A of the monomial curve parametrized by (t^6, t^7, t^{15}). Then A is a Cohen-Macaulay local domain of dimension one. In particular, $A = k[[x, y, z]]/I$ where $I = (y^3 - xz, x^5 - z^2)$ is generated by a regular sequence. Now, $gr_{\mathfrak{m}}(A) \simeq k[x, y, z]/I^*$ where $I^* = (xz, z^2, y^3z, y^6)$. Hence:

(1) A is a domain, but $gr_{\mathfrak{m}}(A)$ is not even reduced;
(2) A is a complete intersection, but $gr_{\mathfrak{m}}(A)$ is not a complete intersection;
(3) A is a 1-dimensional Cohen-Macaulay ring, but depth $gr_{\mathfrak{m}}(A) = 0$.
(4) The Hilbert series of A is

$$HS_A(z) = \frac{1 + 2z + z^2 + z^3 + z^5}{1-z}$$

and A is Cohen-Macaulay, but its Hilbert function is not admissible for a Cohen-Macaulay graded algebra.

Another interesting example ($\dim A = 2$) was given by J. Herzog, M.E. Rossi and G. Valla in [**22**].

Example 1.2. Let $A = k[[x, y, w, t]]/I$ be where $I = (x^3 - y^7, x^2y - xt^3 - w^6)$. Then A is a 2-dimensional complete intersection, $I^* = (x^3, x^2y, x^2t^3, xt^6, x^2w^6, xy^9 - xw^6t^3, xy^8t^3, y^7t^9)$, $\dim gr_{\mathfrak{m}}(A) = 2$, but depth $gr_{\mathfrak{m}}(A) = 0$.

Notice that in the graded case the h-polynomial of a complete intersection is always symmetric, this is no longer true in the local case.

S. Kleiman proved that there is a finite number of admissible Hilbert functions for graded domains with fixed multiplicity and dimension. The analogous of Kleiman's result does not hold in the local case. The following example shows that the class of local domains of dimension two and multiplicity 4 does not have a finite number of Hilbert functions.

Example 1.3. Let $r > 1$ and

$$A_r := k[[x, y, w, t]]/\wp_r$$

where

$$\wp_r = (w^r t^r - xy, x^3 - w^{2r}y, y^3 - t^{2r}x, x^2t^r - y^2w^r)$$

is a prime ideal in $k[[x, y, w, t]]$. The associated graded ring of A_r is the standard graded algebra

$$gr_{\mathfrak{m}}(A_r) = k[x, y, w, t]/(xy, x^3, y^3, x^2t^r - y^2w^r),$$

and

$$HS_{A_r}(z) = \frac{1 + 2z + 2z^2 - z^{r+2}}{(1-z)^2}.$$

Hence, for all r, we have $e_0 = 4$, $d = 2$, but the Hilbert function depends on r.

Nevertheless, V. Srinivas and V. Trivedi in [**53**] and [**54**] proved that the number of Hilbert Functions of Cohen-Macaulay local rings with given multiplicity and dimension is finite (a different proof was given by M.E. Rossi, N.V. Trung and G. Valla in [**42**]). This is a very interesting result and it produces upper bounds on the Hilbert coefficients. If $(A,\mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d and multiplicity e_0, then

$$|e_i| \leq e_0^{3(i!)-i} - 1 \quad \text{for all } i \geq 1$$

(see [**53**] Theorem 1, [**42**], Corollary 4.2). The result had been extended by several authors without assuming the Cohen-Macaulyness of A and in terms of homological degrees.

The problem of characterizing the admissible numerical functions for Cohen-Macaulay local rings is largely open. M. E. Rossi, G. Valla and W. Vasconcelos in [**48**] proved that, if $(A,\mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d and multiplicity e_0, then

$$HS_A(z) \leq \frac{1+(e_0-1)z}{(1-z)^d}$$

and the equality has been characterized. From this result we achieve other bounds, unfortunately very far from being sharp.

After the pionering work of D. G. Northcott in the 50's, several papers were written in order to better understand the Hilbert function of a Cohen-Macaulay local ring in relation with its Hilbert coefficients. Here we shall present some of them.

2. Superficial elements

A fundamental tool in local algebra is the notion of superficial element. This notion goes back to the work of P. Samuel ([**60**] p.296).

Definition 2.1. An element $a \in \mathfrak{m}$, is said to be superficial for $\mathfrak{m}$ if there exists a non-negative integer c such that

$$(\mathfrak{m}^{n+1} : a) \cap \mathfrak{m}^c = \mathfrak{m}^n$$

for all $n \geq c$.

For every $a \in \mathfrak{m}$ and $n \geq c$, $\mathfrak{m}^n$ is contained in $(\mathfrak{m}^{n+1} : a) \cap \mathfrak{m}^c$. It is the other inclusion that makes superficial elements special. It is clear that, if A is Artinian, then every element is superficial, hence, in this section we will assume $\dim A = d > 0$. From the very definition, we deduce that, if a is a superficial element, then $a \in \mathfrak{m} \setminus \mathfrak{m}^2$. We give now equivalent conditions for an element to be superficial. The following development of the theory of superficial elements basically follows Kirby's work in [**31**].

Consider $N := H^0_{\mathfrak{m}}(A) = \{x \in A \mid \exists\, n, \mathfrak{m}^n x = 0_A\}$ the 0-th local cohomology of A with support in $\mathfrak{m}$. Let $G = \mathrm{gr}_{\mathfrak{m}}(A)$ and denote by H the 0-th local cohomology $H^0_{G_+}(G)$ of G with support in $G_+ = \oplus_{i>0}\mathfrak{m}^i/\mathfrak{m}^{i+1}$. Further $Ass(G)$ will denote the set of the associated primes of G.

Theorem 2.2. *Let $a \in \mathfrak{m} \setminus \mathfrak{m}^2$, the following conditions are equivalent:*

1. *a is superficial for $\mathfrak{m}$.*

2. $a^* \notin \bigcup_{i=1}^m \mathfrak{P}_i$ *with* $\mathfrak{P}_i \in Ass(G)$, $\mathfrak{P}_i \neq G_+$.

3. $H :_G a^* = H$.

4. $N : a = N$ *and* $\mathfrak{m}^{j+1} \cap (a) = a\mathfrak{m}^j$ *for all large* j.

5. $(0 :_G a^*)_j = 0$ *for all large* j.

6. $\mathfrak{m}^{j+1} : a = \mathfrak{m}^j + (0 :_A a)$ *and* $\mathfrak{m}^j \cap (0 :_A a) = 0$ *for all large* j.

We notice that, if the residue field $A/\mathfrak{m}$ is infinite, the existence of superficial elements is given by Condition 2. Hence from now on it will be useful to assume that the residue field is infinite. With regard to this, we read the following standard fact which says that our assumption is not restrictive.

PROPOSITION 2.3. *Let* $(A, \mathfrak{m})$ *be a local ring,* $B = A[x]$ *where* x *is an indeterminate and* $\wp = \mathfrak{m}B$*. Then, for every integer* $n \geq 0$,

$$HF_A(n) = HF_{B_\wp}(n).$$

Since A and $B_\wp$ have the same Hilbert function, they have same dimension, multiplicity and embedding dimension. Moreover, since $B_\wp$ is a flat extension of A, if A is Cohen-Macaulay, then $B_\wp$ is Cohen-Macaulay with a larger residue field.

If depth $A > 0$, then every superficial element for $\mathfrak{m}$ is also A-regular and a is superficial for $\mathfrak{m}$ if and only if $\mathfrak{m}^{j+1} : a = \mathfrak{m}^j$ for all large j. Moreover Proposition 2.2 5. says that a is superficial if and only if a^* is an homogeneous *filter-regular* element in G. We refer to [**55**] for the definition and the properties of homogeneous filter-regular elements. As the geometric meaning of superficial elements we refer to the papers by Bondil and Le (see [**4**], [**5**]).

We collect in the following theorem important results on superficial elements. They will be very useful in order to control numerical invariants of A under generic hyperplane sections.

THEOREM 2.4. *Let* a *be a superficial element for* $\mathfrak{m}$ *and let* $d > 0$ *be the dimension of* A*. We have*

1. $\dim(A/aA) = d - 1$

2. a *is* A*-regular* $\iff$ depth $A > 0$

3. $j \geq 1$, depth $A/aA \geq j \iff$ depth $A \geq j + 1$

4. a^* *is* $gr_\mathfrak{m}(A)$*-regular* $\iff$ depth $gr_\mathfrak{m}(A) > 0$

5. *Sally's machine:* $j \geq 1$, depth $gr_{\mathfrak{m}/(a)}(A/(a)) \geq j \iff$ depth $gr_\mathfrak{m}(A) \geq j + 1$

6. $e_j(A) = e_j(A/(a))$ *for every* $j = 0, \ldots, d - 2$

7. $e_{d-1}(A/(a)) = e_{d-1}(A) + (-1)^{d-1}\lambda(0 : a)$

8. a^* *is a* $gr_\mathfrak{m}(A)$*-regular element* $\iff HS_A(z) = \frac{HS_{A/(a)}(z)}{1-z} \iff$ a *is* A*-regular and* $e_d(A) = e_d(A/(a))$.

We remark that if depth $A > 0$, then a is superficial for $\mathfrak{m}$ if and only if $e_j(A) = e_j(A/(a))$ for every $j = 0, \ldots, d - 1$.

We recall that Property 5. is also named *Sally's machine* because it was proved by J. Sally in dimension one and stated in the same paper for higher dimension. It is a very useful trick in inductive arguments and it was proved by S. Huckaba and T. Marley in [**24**]. Property 3. is well known when a is a regular element, but it seems ignored in literature for superficial elements, a proof can be found in [**46**]. The remaining properties can be easily proved by the definition of superficial element and by means of the following result due to B. Singh. First, we recall the definition of the Hilbert-Samuel function

$$HF^1_A(j) := \lambda(A/\mathfrak{m}^{j+1}) = \sum_{n=0}^{j} HF_A(n).$$

In particular $HF^1_A(n) - HF^1_A(n-1) = HF_A(n)$. We will denote by $P_A(X)$ the corresponding polynomial function which is called the Hilbert-Samuel polynomial.

LEMMA 2.5. *Let $a \in \mathfrak{m}$. Then, for every $j \geq 0$,*

$$HF_A(j) = HF^1_{A/aA}(j) - \lambda(\mathfrak{m}^{j+1} : a/\mathfrak{m}^j).$$

In the following we let h be the embedding codimension of A ($embcodim(A)$), namely

$$h := HF_A(1) - d.$$

Remark that if a is a superficial element, then $embcodim(A) = embcodim(A/(a))$.

An usual trick in reducing problems to positive depth is the following. Let $A^{sat} := A/H^0_{\mathfrak{m}}(A)$ where $H^0_{\mathfrak{m}}(A)$ denotes the 0-th local cohomology of A. We remark that $e_0(A^{sat}) = e_0(A)$. If $a \in \mathfrak{m}$ is a superficial element for $\mathfrak{m}$, then $\overline{a} \in A^{sat}$ is a superficial element for $\mathfrak{m}/H^0_{\mathfrak{m}}(A)$.

We relate the Hilbert coefficients of A with those of A^{sat} :

- $e_i(A) = e_i(A^{sat}) \quad 0 \leq i \leq d-1;$
- $e_d(A) = e_d(A^{sat}) + (-1)^d \lambda(H^0_{\mathfrak{m}}(A))$

If A is 1-dimensional, S. Goto and K. Nishida gave a nice interpretation of $\lambda(H^0_{\mathfrak{m}}(A))$ in terms of the first Hilbert coefficient of an ideal generated by a parameter of A, see [**20**] Lemma 2.4. Concerning the problem in higher dimension, very interesting results by L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T. Phuong and W. Vasconcelos were recently proved, see [**19**]. On this topic we point out also [**32**] by M. Mandal and J. Verma.

A sequence of elements $a_1, \ldots, a_r$ ($r \leq d$) is a *superficial sequence* for $\mathfrak{m}$ if for every $j = 1, \ldots, r$ the element $\overline{a_j}$ is superficial for $\mathfrak{m}/(a_1, \ldots, a_{j-1})$.

Let $a_1, \ldots, a_d$ be a (maximal) superficial sequence for $\mathfrak{m}$ and let $J = (a_1, \ldots, a_d)$, then

$$\mathfrak{m}^{n+1} = J\mathfrak{m}^n \text{ for } n \gg 0. \tag{2.1}$$

The above equality says that J is a minimal reduction of $\mathfrak{m}$. If the residue field is infinite, there is a 1-1 correspondence between maximal superficial sequences and minimal reductions (see for example [**26**]). Every minimal reduction J of $\mathfrak{m}$ can be generated by a maximal superficial sequence and, conversely, the ideal generated by a maximal superficial sequence is a minimal reduction of $\mathfrak{m}$.

The above results can be extended from the $\mathfrak{m}$-adic filtrations to the $\mathfrak{q}$-good (or stable) filtrations, where $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal. In particular to the classical $\mathfrak{q}$-adic filtration (see [**47**]).

3. 1-dimensional local Cohen-Macaulay rings

Assume $(A,\mathfrak{m})$ to be a Cohen-Macaulay local ring of dimension one and embedding codimension h. J. Elias (see [**12**]) characterized the admissible Hilbert-Samuel polynomials

$$P_A(X) = e_0(X+1) - e_1$$

for any pair (e_0, e_1) such that $e_0 - 1 \le e_1 \le \binom{e_0}{2} - \binom{h}{2}$. Nevertheless the problem of determining the admissible Hilbert functions of one-dimensional Cohen-Macaulay local rings is still far from being solved. The question has a clear geometric meaning related to singularities of affine curves which are arithmetically Cohen-Macaulay. In the following theorem we collects some classical results due to J. Herzog and R. Waldi, D.G. Northcott and D. Kirby giving easier proofs.

Theorem 3.1. *Let $(A,\mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. For every $j \ge 0$ we set $v_j = \lambda(\mathfrak{m}^{j+1}/a\ \mathfrak{m}^j)$ where a is a superficial element for $\mathfrak{m}$. Then*

(1) $HF_A(j) = e_0 - v_j\ (\le e_0)$.
(2) *If $HF_A(n) = e_0$ for some integer n, then $HF_A(j) = e_0$ for every $j \ge n$.*
(3) $e_1 = \sum_{j\ge 0} v_j$.
(4) $HF_A(j) \ge \min\{e_0, j+1\}$.
(5) *If $HF_A(n) = n+1$ for some integer $n > 0$, then $HF_A(j) = \min\{j+1, e_0\}$ for every $j \ge n$.*
(6) $e_0 - 1 \le e_1 \le \binom{e_0}{2}$.

Proof. Recall that $e_0 = \lambda(A/aA)$. From the diagram

$$\begin{array}{ccc} A & \supset & \mathfrak{m}^{j+1} \\ \cup & & \cup \\ aA & \supset & a\mathfrak{m}^j \end{array}$$

we easily get

$$HF_A(j) = e_0 - v_j \tag{3.1}$$

where $v_j = \lambda(\mathfrak{m}^{j+1}/a\mathfrak{m}^j)$. It is clear that $HF_A(j) \le e_0$ and if $HF_A(n) = e_0$ for some integer n, then $HF_A(j) = e_0$ for every $j \ge n$. since $v_j = 0$ for every $j \ge n$. Moreover for $n \gg 0$, $v_n = 0$. As a consequence, $HF_A^1(n) = \sum_{j=0}^n HF_A(j) = e_0(n+1) - \sum_{j\ge 0} v_j = e_0(n+1) - e_1$. Hence, $e_1 = \sum_{j\ge 0} v_j$ and (1) (2) and (3) are proved.

Now if there exists $j < e_0$ such that $HF_A(j) \le j$, then by Macaulay's Theorem (see e.g. Theorem 1.3 [**56**]), we should have $HF_A(n) \le j < e_0$ for every n, which is a contradiction because $HF_A(n) = e_0$ for large n. Hence (4) and (5) follow. We remark that by (3), $e_1 \ge v_0 = \lambda(\mathfrak{m}/(a)) = e_0 - 1$ and hence

$$e_0 - 1 \le e_1 = \sum_{j\ge 0} v_j = \sum_{j\ge 0}(e_0 - HF_A(j)) \le \sum_{j=0}^{e_0-1}(e_0 - (j+1)) = \binom{e_0}{2}.$$

□

We remark that Theorem 3.1 (5) was also proved by J. Herzog and V. Waldi (see [**23**]) by using different methods.

The following statements follow by Theorem 3.1 and they underline the general philosophy that extremal values of the Hilbert coefficients (with respect to some bound) answer to the whole Hilbert function. The proof is an easy exercise.

Exercise 3.2. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one and multiplicity e_0. The following facts hold:

(1) $e_1 = \binom{e_0}{2} \iff HS_A(z) = \frac{\sum_{i=0}^{e_0-1} z^i}{1-z}$

(2) $e_1 = \binom{e_0}{2} - 1 \iff HS_A(z) = \frac{1+2z+\sum_{i=2}^{e_0-1} z^i}{1-z}$

It is clear that, if we introduce more invariants of A, we can get more precise bounds. For example if we involve the embedding codimension of A, $h = HF_A(1) - 1$, then we have $v_1 = \lambda(\mathfrak{m}^2/a\mathfrak{m}) = \lambda(\mathfrak{m}/(a)) + \lambda((a)/a\mathfrak{m}) - \lambda(\mathfrak{m}/\mathfrak{m}^2) = e_0 - 1 - h$. Hence

$$e_1 \geq v_0 + v_1 = 2e_0 - (h+2). \tag{3.2}$$

By using sophisticated devices introduced by Matlis in [**33**], J. Elias proved that

$$e_1 \leq \binom{e_0}{2} - \binom{h}{2} \tag{3.3}$$

and, for every triple (e_0, e_1, h) satisfying the inequalities (3.2) and (3.3), one can produce an affine curve with such invariants (see [**12**]).

In a more general setting, a different proof of (3.3) was given by G. Valla and the author, by using the Ratliff-Rush filtration (see [**45**]). In both proofs the techniques are quite complicate and we do not present here them. Nevertheless, the inequality (3.3) is an easy exercise assuming an extra (unnecessary) condition.

Exercise 3.3. Let $(A, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring, with embedding dimension h and multiplicity e_0. *Assume that the Hilbert function of A is not decreasing*, i.e. $HF_A(n+1) \geq HF_A(n)$ for every n. Then $e_1 \leq \binom{e_0}{2} - \binom{h}{2}$.

If $HF_A(n+1) \geq HF_A(n)$ for every n, we say that the Hilbert function is non-decreasing. It is a natural question to ask whether the Hilbert function of a one-dimensional Cohen-Macaulay ring is not decreasing. Clearly, this is the case if $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay, but this is not a necessary requirement (see Example 1.1).

Unfortunately, it can happen that $HF_A(2) = \mu(\mathfrak{m}^2) < HF_A(1) = \mu(\mathfrak{m})$. The first example was given by J. Herzog and R. Waldi in 1975. If we consider the semigroup ring

$$A = k[[t^{30}, t^{35}, t^{42}, t^{47}, t^{148}, t^{153}, t^{157}, t^{169}, t^{181}, t^{193}]]$$

of multiplicity $e_0 = 30$ and embedding dimension $HF_A(1) = 10$, we have $HF_A(2) = 9 < HF_A(1)$. In 1980 F. Orecchia proved that, for all embedding dimension $b \geq 5$, there exists a reduced one-dimensional local ring of embedding dimension b and decreasing Hilbert function. L. Roberts in 1982 built ordinary singularities with locally decreasing Hilbert function and embedding dimension at least 7.

We recall that Gupta and Roberts proved that there exists a one-dimensional Cohen-Macaulay local ring of embedding dimension 4 and multiplicity 32 with locally decreasing Hilbert function.

It is interesting to ask what happens if the embedding dimension is < 4. It is clear that if $HF_A(1) = 1$, then A is regular. If $HF_A(1) = 2$, then by Theorem 3.1 (5) we conclude that $HF_A(n) = \min\{n+1, e_0\}$, hence it is non-decreasing. The only unknown case seems to be $HF_A(1) = 3$.

In 1978 J. Sally stated the following conjecture.

Conjecture: If A is a one-dimensional Cohen-Macaulay local ring with embedding dimension three, then HF_A is non-decreasing.

J. Elias gave a positive answer to the problem stated by J. Sally for the equicharacteristic rings of embedding dimension three (see [**13**]). Following the geometrical idea by J. Elias, we present here an elementary, but tricky proof.

We assume $A = R/I$ where $R = k[[x_1, \dots, x_n]]$ with maximal ideal $\mathfrak{n} = (x_1, \dots, x_n)$. We may assume k infinite and $I \subseteq \mathfrak{n}^2$. Denote by $t(I)$ the initial degree of I, namely

$$t(I) = \min\{j \ : \ HF_A(j) \neq \binom{n+j-1}{j}\}.$$

It is well known that $gr_{\mathfrak{m}}(A) \simeq gr_{\mathfrak{n}}(R)/I^* = k[X_1, \dots, X_n]/I^*$ where I^* is the ideal generated by the initial forms of the elements of I.
Let $P = (a_1, \dots, a_n) \neq (0, \dots, 0)$ in k^n and, if $a_i \neq 0$, denote by $L = L(P)$ the ideal of R generated by the elements $a_i x_j - a_j x_i$ for $j = 1, \dots, n$. Remark that L^* is the ideal of $k[X_1, \dots, X_n]$ generated by the elements $a_i X_j - a_j X_i$. In particular L^* is the defining ideal of the point $P = (a_1, \dots, a_n) \in \mathbf{P}_k^{n-1}$, we will write $L^* = L^*(P)$. With the above notation we can prove the following result.

Lemma 3.4. *Let $A = R/I$ be a one-dimensional Cohen-Macaulay local ring. There exists $P = (a_1, \dots, a_n) \in k^n$ such that if we let $J = I \cap L(P)$ then*

$$HF_{R/J}(j) = HF_A(j) + 1 \quad \textit{for every} \quad j \geq t(I).$$

Proof. Denote by $I^*_{<t(I)>}$ the ideal generated by the homogeneous part of degree $t(I)$ of I^*. First we prove that there exists $P = (a_1, \dots, a_n) \neq (0, \dots, 0) \in k^n$ such that $I^*_{<t(I)>} \not\subseteq L^*(P)$. Let $X = \{P_1, \dots, P_s\}$ be a set of $s \geq \binom{n+t(I)-1}{t(I)}$ points of $\mathbf{P}_k^{n-1}$ with maximal Hilbert function (it is well known that for every integers $n, t(I)$ such X exists). Since the ideal $I(X) = \cap_{i=1}^s L^*(P_i)$ has initial degree $t(I)+1$, there exists $P \in X$ such that $I^*_{<t(I)>} \not\subseteq L^*(P)$. We may assume $P = (a_1, \dots, a_n)$ with $a_1 \neq 0$. Hence $L = L(P) = (a_2 x_1 - a_1 x_2, \dots, a_n x_1 - a_1 x_n)$ and $\mathfrak{n} = (x_1) + L$. From $I^*_{<t(I)>} \not\subseteq L^*(P)$ we deduce that there exists a generator f of I such that $f = x_1^{t(I)} + \alpha$ with $\alpha \in L \cap \mathfrak{n}^{t(I)}$. It follows

$$I + L = (x_1^{t(I)}) + L. \tag{3.4}$$

In fact $I \subseteq \mathfrak{n}^{t(I)} \subseteq (x_1^{t(I)}) + L$. On the other hand we remark that $x_1^{t(I)} = f - \alpha \in I + L$. Hence, by (3.4), $\dim_k R/(I+L) = t(I)$. Now we claim that, for every $j \geq t(I) - 1$, we have

$$\mathfrak{n}^{j+1} + J = (\mathfrak{n}^{j+1} + L) \cap (\mathfrak{n}^{j+1} + I). \tag{3.5}$$

By a repeated use of the "modular law":

$$(\mathfrak{n}^{j+1} + L) \cap (\mathfrak{n}^{j+1} + I) = \mathfrak{n}^{j+1} + L \cap (\mathfrak{n}^{j+1} + I) = \mathfrak{n}^{j+1} + L \cap ((x_1^{j+1}) + I)$$

and the conclusion still follows by the modular law since, by (3.4), $x_1^{j+1} \in I + L \cap \mathfrak{n}^{j+1}$ for every $j \geq t(I) - 1$. Using (3.5), it is well defined the following exact sequence of R-modules (of finite length):

$$0 \to R/(\mathfrak{n}^{j+1} + J) \to R/(\mathfrak{n}^{j+1} + I) \oplus R/(\mathfrak{n}^{j+1} + L) \to R/(\mathfrak{n}^{j+1} + I + L) \to 0.$$

Since the length is an additive function:

$$HF^1_{R/J}(j) = HF^1_{R/I}(j) + HF^1_{R/L}(j) - t(I) = HF^1_{R/I}(j) + j - t(I).$$

It follows $HF_{R/J}(j) = HF^1_{R/J}(j) - HF^1_{R/J}(j-1) = HF_{R/I}(j) + 1$ for every $j - 1 \geq t(I) - 1$, hence $j \geq t(I)$.

□

The above Lemma has a clear geometric meaning. It describes the behavior of the Hilbert function by adding to the curve corresponding to $Spec(R/I)$, a straight line $Spec(R/L)$.

We remark that $R/I \cap L$ is still one-dimensional and Cohen-Macaulay (primes avoidance) of the same embedding dimension of R/I.

THEOREM 3.5. *Let $A = R/I$ be a one-dimensional Cohen-Macaulay local ring of embedding dimension three. Then the Hilbert function of A is non-decreasing.*

PROOF. Assume that there exists n_0 such that $HF_A(n_0) > HF_A(n_0 + 1)$. By a repeated use of Lemma 3.4, we may assume that $n_0 = t(I) - 1$. It follows

$$\mu(I) \geq \binom{t(I)+2}{2} - HF_A(t(I)) \geq \binom{t(I)+2}{2} - HF_A(t(I) - 1) + 1 =$$

$$= \binom{t(I)+2}{2} - \binom{t(I)+1}{2} + 1 = t(I) + 2.$$

On the other hand, from the Hilbert-Burch Theorem, we deduce that $\mu(I) \leq t(I) + 1$, a contradiction. □

We characterize other interesting classes of one-dimensional Cohen-Macaulay local rings with non-decreasing Hilbert function. By Theorem 3.1 (1), we deduce that for every superficial element $a \in \mathfrak{m}$ we have:

$$e_0 = \mu(\mathfrak{m}) + \lambda(\mathfrak{m}^2/a\mathfrak{m}) = h + 1 + \lambda(\mathfrak{m}^2/a\mathfrak{m}) \tag{3.6}$$

where h is the embedding codimension, hence

$$e_0 \geq h + 1 = \mu(\mathfrak{m}) = HF_A(1)$$

which is the one-dimensional version of the well-known Abhyankar's inequality (see [**1**]). We prove now that the Hilbert function of a one-dimensional Cohen-Macaulay local ring of small multiplicity is non-decreasing.

THEOREM 3.6. *Let $A = R/I$ be a one-dimensional Cohen-Macaulay local ring of multiplicity $e_0 \leq h + 3$. Then the Hilbert function of A is non-decreasing.*

Proof. Let a be a superficial element for $\mathfrak{m}$ and consider the corresponding Artinian reduction $B = A/aA$. If we assume $HF_B(3) = 0$, equivalently $\mathfrak{m}^3 \subseteq a\mathfrak{m}$, then $\lambda(\mathfrak{m}^2/a\mathfrak{m}) = \lambda(\mathfrak{m}^2/a\mathfrak{m} + \mathfrak{m}^3) \leq 2$ by (3.6). Consider the standard graded k-algebra

$$T = \oplus_{i \geq 0} \mathfrak{m}^i/(a\ \mathfrak{m}^{i-1} + \mathfrak{m}^{i+1}).$$

Because $HF_T(2) = \lambda(\mathfrak{m}^2/a\mathfrak{m}) \leq 2$, by Macaulay's Theorem we get $HF_T(i) \leq 2$ for every $i \geq 2$. But if $HF_T(i_0) \leq 1$ for some $i_0 \geq 2$, then $HF_T(i) \leq 1$ for every $i \geq i_0$. Hence HF_T is not locally increasing and we conclude by Theorem 3.1 (1), indeed for $i \geq 2$ $HF_A(i) = e_0 - \lambda(\mathfrak{m}^{i+1}/a\mathfrak{m}^i) = e_0 - HF_T(i)$.
Assume now $HF_B(3) \neq 0$, since $e_0 = \sum_{i \geq 0} HF_B(i) = h+3$, necessarily $HF_B(2) = HF_B(3) = 1$. Hence $\lambda(\mathfrak{m}^2/a\mathfrak{m}) = 2$ by (3.6) and $\lambda(\mathfrak{m}^2/a\mathfrak{m}+\mathfrak{m}^3) = HF_B(2) = 1$. We deduce that $\mathfrak{m}^2/a\mathfrak{m}$ is a cyclic module (it is not a k-vector space) and there exists $w \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathfrak{m}^2 = a\mathfrak{m} + (w^2)$. In fact if $w^2 \in a\mathfrak{m}$ for every $w \in \mathfrak{m} \setminus \mathfrak{m}^2$, then it is easy to prove that $\mathfrak{m}^3 \subseteq a\mathfrak{m}$. Therefore for every $j \geq 1$

$$\mathfrak{m}^{j+1} = a\mathfrak{m}^j + (w^{j+1}).$$

Since the map $\mathfrak{m}^{j+1}/a\mathfrak{m}^j \stackrel{\cdot w}{\to} \mathfrak{m}^{j+2}/a\mathfrak{m}^{j+1}$ is an epimorphism, we conclude because

$$HF_A(j) = e_0 - \lambda(\mathfrak{m}^{j+1}/a\mathfrak{m}^j) \leq HF_A(j+1) = e_0 - \lambda(\mathfrak{m}^{j+2}/a\mathfrak{m}^{j+1})$$

for every j.

□

The above situation cannot be extended to multiplicity $e_0 = h+4$. S. Molinelli and G. Tamone produced the following example with multiplicity $e_0 = h+4$ and locally decreasing Hilbert function. Consider

$$A = k[[t^{13}, t^{19}, t^{24}, t^{44}, t^{49}, t^{54}, t^{55}, t^{59}, t^{60}, t^{66}]].$$

Then $HF_A(2) = 9 < HF_A(1) = 10$.

Starting from one-dimensional Cohen-Macaulay local rings with locally decreasing Hilbert functions, it is possible to produce examples with several peaks and valleys. The technique had been pointed out by J. Elias. Let $A = k[[x_1, \ldots, x_r]]/I$ and $B = k[[y_1, \ldots, y_s]]/J$ be one dimensional Cohen-Macaulay local rings and consider the ideal

$$K = [I + (y_1, \ldots, y_s)] \cap [J + (x_1, \ldots, x_r)] \subseteq k[[x_1, \ldots, x_r, y_1, \ldots, y_s]] = C$$

Then

$$HF_{C/K}(n) = HF_A(n) + HF_B(n)$$

for every $n \geq 1$. The crucial point is that

$$K^* = [I + (y_1, \ldots, y_s)]^* \cap [J + (x_1, \ldots, x_r)]^* = [I^* + (y_1^*, \ldots, y_s^*)] \cap [J^* + (x_1^*, \ldots, x_r^*)].$$

Several computations lead us to think that these pathologies cannot be realized if the local ring is Gorenstein. We state the following problem:

Problem 3.7. *Is the Hilbert function of a Gorenstein local ring of dimension one not decreasing?*

The problem is open even if we consider complete intersections. Very partial results have been proved. Puthenpurakal gave a positive answer in codimension two case (see [**37**]). Some results appeared for Gorenstein monomial curves in $\mathbf{A}^4$ (see [**2**]).

4. Hilbert coefficients and classical bounds

In this section we will extend the interest to higher dimensions. After the pioneering work done by Northcott (see [**35**]), several efforts have been made also to better understand the Hilbert function of a d-dimensional Cohen-Macaulay local ring $(A,\mathfrak{m})$ starting from the asymptotic information given by the Hilbert coefficients. J. Sally carried on Northcott's work (see for example [**49**], [**50**]) by proving interesting results on this topic and by asking challenging questions as well. Several improvements along this line have been obtained in the last years (J. Elias, C. Huneke, S. Itoh, A. Ooishi, C. Polini, T. Puthenpurakal, M. E. Rossi, G. Valla, W. Vasconcelos, J. Verma, etc.), often extending the framework to Hilbert functions associated to an $\mathfrak{m}$-primary ideal or, more in general, to a stable filtration of A-modules (see [**47**]).

The first Hilbert coefficient e_0 is the multiplicity and, due to its geometric meaning, has been studied very deeply. From the algebraic point of view $e_0 = \lambda(A/J)$ where J is the ideal generated by a maximal superficial sequence, in particular a minimal reduction of $\mathfrak{m}$. The other coefficients are not as well understood, either geometrically or in terms of how they are related to algebraic properties of the local ring.

We present here elementary proofs of two classical bounds due to Abhyankar and Northcott respectively (see [**1**] and [**35**]). As usual, we denote by e_i the Hilbert coefficients and by h the embedding codimension of A, i.e. $h = HF_A(1) - d = \mu(\mathfrak{m}) - d$. We recall that, since $(A,\mathfrak{m})$ is Cohen-Macaulay, every superficial sequence is also a regular sequence in A.

THEOREM 4.1. *(Abhyankar's inequality) Let $(A,\mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring. Then*

$$e_0 \geq h + 1.$$

PROOF. Since e_0 and h do not change modulo a superficial sequence (see Theorem 2.4 (7)), we may assume $d = 0$. Then

$$HS_A(z) = 1 + hz + \cdots + h_s z^s$$

where $s \geq 1$ and $h_i \geq 0$. Thus $e_0 = 1 + h + \cdots + h_s \geq 1 + h$. □

The above result shows that $e_0 = 1$ if and only if $h = \mu(\mathfrak{m}) - d = 0$, hence A is regular. We remark that being A Cohen-Macaulay is essential. Let $A = k[[x,y]]/(x^2, xy)$, then A is not Cohen-Macaulay and $e_0 = 1 < h + 1 = 2$. Hence we deduce that the standard graded k-algebra $G = k[x,y]/(x^2, xy)$ cannot be the associated graded ring of a Cohen-Macaulay local ring.

Theorem 4.1 extends to the local Cohen-Macaulay rings the well known lower bound for the degree of a reduced and irreducible non-degenerate variety X in $\mathbb{P}^n$:

$$\deg X \geq \operatorname{codim} X + 1.$$

The varieties for which the bound is attained are called varieties of minimal degree and they are completely classified. In particular, they are always arithmetically Cohen-Macaulay.

In the following result, the first inequality has been proved by J. Elias and G. Valla in [**15**], the second is a classical result due to Northcott (see [**35**]).

THEOREM 4.2. *Let* $(A, \mathfrak{m})$ *be a d-dimensional Cohen-Macaulay local ring. Then*

$$e_1 \geq 2e_0 - h - 2 \geq e_0 - 1.$$

PROOF. By Theorem 2.4, parts 6. and 7. and the fact that the embedding codimension h does not change modulo superficial elements, we may assume $d = 1$. Then the result follows by (3.2). The second inequality is a consequence of Theorem 4.1. □

THEOREM 4.3. *Let* $(A, \mathfrak{m})$ *be a d-dimensional Cohen-Macaulay local ring. The following facts are equivalent:*

(1) $e_0 = h + 1$

(2) $\mathfrak{m}^{n+1} = J\mathfrak{m}^n$ *for every* $n \geq 1$ *and for every minimal reduction* J *of* $\mathfrak{m}$.

(3) $HS_A(z) = \frac{1+hz}{(1-z)^d}$

(4) $e_1 = e_0 - 1$

PROOF. Let J be a minimal reduction of $\mathfrak{m}$ (or equivalently the ideal generated by a maximal superficial sequence). From the diagram

$$\begin{array}{ccc} \mathfrak{m} & \supset & \mathfrak{m}^2 \\ \cup & & \cup \\ J & \supset & J\mathfrak{m} \end{array}$$

we get $\lambda(\mathfrak{m}/J) = e_0 - 1 = HF_A(1) - d + \lambda(\mathfrak{m}^2/J\mathfrak{m})$. Hence, if $e_0 = h+1$, then $\mathfrak{m}^2 = J\mathfrak{m}$ and therefore (1) implies (2). If we assume (2), by the well-known Valabrega-Valla criterion, we have that $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay. Hence by Theorem 2.4, part 8., we may reduce the computation of HS_A to that of the Artinian reduction $B = A/J$. Since the embedding codimension does not change and $(\mathfrak{m}/J)^2 = 0$ we get (3). The implication (3) $\Longrightarrow$ (4) is clear. If we assume (4), by Theorem 4.2, we get $e_1 = 2e_0 - h - 2 = e_0 - 1$, therefore $e_0 = h + 1$.

□

Hence the next step will be the study of the Hilbert functions of Cohen-Macaulay local rings of almost minimal multiplicity, that is $e_0 = h + 2$. The problem is considerably more difficult. Also from the geometric point of view, varieties satisfying $deg\ X = codim\ X + 2$ are not necessarily arithmetically Cohen-Macaulay.

We propose in the following exercise the easy part of the general result.

EXERCISE 4.4. Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring. The following facts are equivalent:

(1) $e_0 = h + 2$ and $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay
(2) $HS_A(z) = \frac{1+hz+z^2}{(1-z)^d}$
(3) $e_1 = e_0$

In the above exercise the Cohen-Macaulyness of $gr_{\mathfrak{m}}(A)$ is essential. In fact if we consider $A = k[[t^4, t^5, t^{11}]]$, then $e_0 = 4 = h+2$, but $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay and $HS_A(z) = \frac{1+hz+z^3}{(1-z)}$.

We come now to a result proved by G. Valla and the author [**45**], and independently by Wang [**59**], which gives a positive answer to a longstanding conjecture stated by J. Sally in [**50**].

THEOREM 4.5. *Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring. The following facts are equivalent:*

(1) $e_0 = h + 2$

(2) $HS_A(z) = \frac{1+hz+z^s}{(1-z)^d}$ *for some integer* $2 \le s \le h+1$.

Further, if either of the above conditions holds, then $\operatorname{depth} gr_{\mathfrak{m}}(A) \ge d-1$.

Sally proved the above result in dimension one and, for every integer s, she gave an example:

$$A_2 = k[[t^e, t^{e+1}, t^{e+3}, \dots, t^{2e-1}]]$$
$$A_s = k[[t^e, t^{e+1}, t^{e+s+1}, t^{e+s+2}, \dots, t^{2e-1}, t^{2e+3}, t^{2e+4}, \dots, t^{2e+s}]],\ 3 \le s \le e-2$$
$$A_{e-1} = k[[t^2, t^{e+1}, t^{2e+3}, t^{2e+4}, \dots, t^{3e-1}]].$$

Sally's conjecture can be proved by reducing the problem to the two-dimensional case. In spite of the fact that in the one-dimensional case the proof of Theorem 4.5 is very easy, in dimension two the result is complicated and a crucial tool in the proof given in [**45**] involves the use of the Ratliff-Rush filtration (see [**38**]) which we recall below.

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring. For every $n \ge 0$ we consider the chain of ideals

$$\mathfrak{m}^n \subseteq \mathfrak{m}^{n+1} : \mathfrak{m} \subseteq \mathfrak{m}^{n+2} : \mathfrak{m}^2 \subseteq \cdots \subseteq \mathfrak{m}^{n+k} : \mathfrak{m}^k \subseteq \cdots$$

This chain stabilizes at an ideal which we will denote by

$$\widetilde{\mathfrak{m}^n} := \bigcup_{k \ge 1} (\mathfrak{m}^{n+k} : \mathfrak{m}^k).$$

Hence there exists a positive integer t, depending on n, such that $\widetilde{\mathfrak{m}^n} = \mathfrak{m}^{n+k} : \mathfrak{m}^k$ for every $k \ge t$. It is clear that $\widetilde{\mathfrak{m}^0} = A$ and, for every non negative integers i and j,

$$\mathfrak{m}^i \subseteq \widetilde{\mathfrak{m}^i},\quad \widetilde{\mathfrak{m}^i}\widetilde{\mathfrak{m}^j} \subseteq \widetilde{\mathfrak{m}^{i+j}},\quad \widetilde{\mathfrak{m}^{i+1}} \subseteq \widetilde{\mathfrak{m}^i}.$$

Furthermore, if a is superficial, for every $i \ge 0$ we have

$$\widetilde{\mathfrak{m}^{i+1}} : a = \widetilde{\mathfrak{m}^i}.$$

It is easy to prove that if depth $A > 0$, then $\widetilde{\mathfrak{m}^i} = \mathfrak{m}^i$ for large integers i. If J is an ideal generated by a maximal superficial sequence, we may define, for every $i \ge 0$,

$$\sigma_i := \lambda(\widetilde{\mathfrak{m}^{i+1}} / J\widetilde{\mathfrak{m}^{i+1}}).$$

For example $\sigma_0 = e_0 - 1$. With the above notation one can prove the following equalities, proved by S. Huckaba and T. Marley (see [**24**]) in the case $\dim A \le 2$.

$$e_1 = \sum_{i \ge 0} \sigma_i \quad \text{and} \quad e_2 = \sum_{i \ge 1} i\sigma_i \tag{4.1}$$

The main tool is to consider the graded k-algebra $\widetilde{G} := \oplus_{i\geq 0}\widetilde{\mathfrak{m}^i}/\widetilde{\mathfrak{m}^{i+1}}$. The Hilbert polynomial of $\widetilde{G}$ is the Hilbert polynomial of A and the advantage is that $\widetilde{G}$ has positive depth even if $gr_{\mathfrak{m}}(A)$ has depth zero.

The crucial point in the proof of Theorem 4.5 is to show that, for every Cohen-Macaulay local ring of dimension at most two, the reduction number $r(\mathfrak{m})$ of $\mathfrak{m}$ is bounded by the following linear function of e_1 and e_0

$$r(\mathfrak{m}) \leq e_1 - e_0 + 2,$$

equivalently we have

$$\mathfrak{m}^{e_1-e_0+3} = J\mathfrak{m}^{e_1-e_0+2}$$

for every minimal reduction J of $\mathfrak{m}$ (see [**39**]).

We do not know if the above equality can be extended to higher dimensions. More in general, it is natural to ask the following problem

PROBLEM 4.6. *Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Does there exist a linear function $f(e_0, e_1, \dots, e_{d-1}, d)$ such that the reduction number of $\mathfrak{m}$ is bounded by $f(e_0, e_1, \dots, e_{d-1}, d)$?*

Using (4.1), we investigate the successive Hilbert coefficients and we present an easy proof of a lower bound on e_2 proved by J. Sally in [**52**].

PROPOSITION 4.7. *Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. Then*

$$e_2 \geq e_1 - e_0 + 1 \geq 0.$$

PROOF. By Theorem 2.4 we prove the result in the case $d = 2$. Then by (4.1) we have

$$e_2 = \sum_{i\geq 1} i\sigma_i = \sum_{i\geq 0} \sigma_i - \sigma_0 + \sum_{i\geq 1}(i-1)\sigma_i \geq e_1 - e_0 + 1.$$

By Theorem 4.2 it follows that $e_2 \geq 0$. □

Clearly, if $e_2 = 0$, then $e_2 = e_1 - e_0 + 1$ and $e_1 = e_0 - 1$. Hence, by Theorem 4.3, the Hilbert function of A is known and $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay. The following example (due to H. J. Wang) shows that the only equality $e_2 = e_1 - e_0 + 1$ does not force $gr_{\mathfrak{m}}(A)$ to be Cohen-Macaulay.

EXAMPLE 4.8. Consider the two-dimensional Cohen-Macaulay local ring

$$A = k[\![x, y, t, u, v]\!]/(t^2, tu, tv, uv, yt - u^3, xt - v^3)$$

with maximal ideal $\mathfrak{m}$. We have

$$HS_A(z) = \frac{1 + 3z + 3z^3 - z^4}{(1-z)^2}.$$

Hence one has $e_0 = h + 3 = 6$, $e_1 = 8, e_2 = 3 = e_1 - e_0 + 1$. The associated graded ring $gr_{\mathfrak{m}}(A)$ has depth zero. Notice that $e_3 < 0$.

It seems that the normality of the ideal $\mathfrak{m}$ (i.e. $\mathfrak{m}^n$ is integrally closed for every n) yields non trivial consequences on the Hilbert coefficients of A and, ultimately, on depth $gr_{\mathfrak{m}}(A)$ and the Hilbert function.

In [**11**] A. Corso, C. Polini and M.E. Rossi proved the following result.

THEOREM 4.9. *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* d. *Assume* $\mathfrak{m}$ *is a normal ideal and* $e_2 = e_1 - e_0 + 1$, *then*

$$HS_A(z) = \frac{1 + hz + (e_0 - h - 1)z^2}{(1-z)^d} \quad \text{and} \quad \mathfrak{m}^3 = J\mathfrak{m}^2$$

for every minimal reduction J *of* $\mathfrak{m}$. *In particular* $gr_{\mathfrak{m}}(A)$ *is Cohen-Macaulay.*

This result gives a positive answer to a question raised by G. Valla in [**56**]. The key point is a theorem proved by S. Itoh on the normalized Hilbert coefficients of ideals generated by a system of parameters (see [**29**], [**30**]).

Unfortunately, the positivity of the Hilbert coefficients stops with e_2. Indeed, M. Narita showed that e_3 can be negative (see also Example 4.8). However, a remarkable result of S. Itoh says that $e_3 \geq 0$ provided $\mathfrak{m}$ is normal (see [**29**]). If equality holds, then $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay. A nice proof of this result was also given by S. Huckaba and C. Huneke in [**25**] where the ideal is assumed to be asymptotically normal. A different proof has been also produced by A. Corso, C. Polini and M.E. Rossi by using techniques developed in this survey (see Theorem 4.1. in [**11**]).
As far as we know there are not negative answers to the following natural question.

PROBLEM 4.10. *If* $\mathfrak{m}$ *is normal, then is* $e_i \geq 0$ *for every* i *?*

The previous problem is related to the asymptotic behavior of the associated graded ring of the powers of the maximal ideal and it has some relation with the Grauert-Riemenschneider Vanishing Theorem. For interesting questions related to the Hilbert coefficients of normal ideals we refer to [**29**] and [**30**].

Finally, we remark that if $e_0 = h + 3$, then Example 4.8 shows that it is no longer true that $\operatorname{depth} gr_{\mathfrak{m}}(A) \geq d - 1$. J. Sally proved that if A is Gorenstein and $e_0 = h + 3$, then $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay (see [**51**]). Later M.E. Rossi and G. Valla (see [**44**]) proved the following result

THEOREM 4.11. *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* d *and Cohen-Macaulay type* τ. *If* $e_0 = h + 3$ *and* $\tau < h$, *then*

$$HS_A(z) = \frac{1 + hz + z^2 + z^s}{(1-z)^d}$$

where $2 \leq s \leq \tau + 2$. *In particular* $\operatorname{depth} gr_{\mathfrak{m}}(A) \geq d - 1$.

We state the following problem.

PROBLEM 4.12. *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* d *and multiplicity* $e_0 = h + 3$. *Then is* $\operatorname{depth} gr_{\mathfrak{m}}(A) \geq d - 2$?

Notice that if $e_0 = h + 3$, then $\tau \leq h + 1$, hence it remains to explore the cases $\tau = h, h + 1$.

Interesting problems on the Hilbert functions also come from the Artinian case, under the assumption that A is Gorenstein or, more in general, level. In the next section we present an interesting approach by using Macaulay's Inverse System.

5. Hilbert Functions of Artinian Gorenstein rings

Let $(A, \mathfrak{m}, k)$ be an Artinian local ring. The socle-degree s of A is the largest integer for which $\mathfrak{m}^s \neq 0$. The local ring A is s-level of type τ if

$$0 : \mathfrak{m} = \mathfrak{m}^s \quad \text{and} \quad \dim_k \mathfrak{m}^s = \tau$$

A is Gorenstein if A has type 1.

Since $HF_A(n) = 0$ for $n > s$, we will write the Hilbert function of the Artinian local ring A as a finite sequence of integers $HF_A = \{HF_A(0), \dots, HF_A(s)\}$.

Very little is known about the Hilbert function of A, if we assume that A is Gorenstein or level. We recall that, in the graded case, the Hilbert function of an Artinian Gorenstein standard k-algebra is symmetric, but this is no longer true if we assume that A is local. For instance, consider the complete intersection $I = (xy, x^2 - y^3) \in R = k[[x, y]]$, then $A = R/I$ has Hilbert function $\{1, 2, 1, 1\}$ which is not symmetric.

It is interesting to recall that, however we fix an integer $e \geq 4$, there exists an ideal $I = (f, g)$ which is a complete intersection of two elements of valuation two in R with multiplicity $e_0(R/I) = e$ (for example see [**3**]). In the corresponding homogeneous setting the multiplicity would be 4!

In the following we will consider numerical functions $H = \{1, h_1, h_2, \dots, h_s\}$, h_i positive integers, which verify Macaulay's theorem and our aim is to characterize those which are admissible for an Artinian Gorenstein local ring. The following nice result has been proved by several authors, among others by A. Iarrobino, S. Kotari, S. Goto et al.

Theorem 5.1. *A numerical function $H = \{1, 2, h_2, \dots, h_s\}$, h_i positive integers, is admissible for an Artinian Gorenstein local ring $A = k[[x, y]]/I$ if and only if*

$$|h_{i+1} - h_i| \leq 1$$

for every $i = 1, \dots, s$ (consider $h_1 = 2$ and $h_i = 0$ if $i > s$).

An interesting generalization to level algebras was presented by V. Bertella (see [**3**]). She proved that a numerical function $H = \{1, 2, h_2, \dots, h_s\}$ is admissible for a s-level Artinian local ring of type τ if and only if

$$|h_{i+1} - h_i| \leq \tau.$$

A very short proof of the above results was recently given by M.E. Rossi and L. Sharifan as a consequence of a result on the minimal free resolutions of local rings of given Hilbert Function (see [**41**]).

A characterization of the Hilbert functions of Artinian Gorenstein local rings of codimension three ($H = \{1, 3, h_2, \dots, h_s\}$) is still an open problem, even if we assume $A = k[[x, y, z]]/I$ where I is a complete intersection.

Another motivation concerning the study of Artinian local rings comes from recent papers by J. Elias and G. Valla (see [**16**], [**17**]), by G. Casnati and R. Notari (see [**7**], [**8**]), by D.A. Cartwright, D. Erman, M. Velasco and B. Viray (see [**6**]) and by B. Poonen (see [**36**]), where the authors studied the classification up to isomorphisms of the Artinian Gorenstein k-algebras of a given Hilbert function.

For example an Artinian Gorenstein local ring A with Hilbert function $\{1, 2, 2, 1\}$ allows only two models, namely those corresponding to the ideals $I = (x^2, y^3)$ and $I = (xy, x^3 - y^3)$ and both are homogneous. But if we move to the next case with symmetric Hilbert function $\{1, 2, 2, 2, 1\}$ we will see that one has three different models, namely two ideals which are homogeneous $I = (x^2, y^4)$, $I = (xy, x^4 - y^4)$ and one which is not homogeneous, the ideal $I = (x^4 + 2x^3y, y^2 - x^3)$. It is interesting to say that there is a finite number of isomorphism classes of Artinian Gorenstein algebras of multiplicity ≤ 9. The first case with a 1-dimensional family of models corresponds to $e_0 = 10$ and Hilbert function $\{1, 2, 2, 2, 1, 1, 1\}$. This case has been studied by Elias and Valla in [**17**].

This classical problem has an important motivation related to the study of the Hilbert scheme $Hilb_d(P_k^N)$ parametrizing scheme of dimension 0 and fixed degree d in P_k^N. Since it is known that any zero-dimensional Gorenstein scheme of degree d can be embedded as an arithmetically Gorenstein non-degenerate subscheme in P_k^{d-2}, it is natural to study the open locus

$$Hilb_d^{aG}(P_k^{d-2}) \subseteq Hilb_d(P_k^{d-2}).$$

The scheme $Hilb_d^{aG}(P_k^{d-2})$ has a natural stratification which reduce the problem to understand the intrinsic structure of Artinian Gorenstein k-algebras of degree d. Since such an algebra is the direct sum of local, Artinian, Gorenstein k-algebras of degree at most d, it is natural to begin with the inspection of some elementary bricks.

The aim of this section is to investigate these problems by means of the Inverse System. In 1916 Macaulay established a one-to-one correspondence between Gorenstein Artin algebras and suitable polynomials. This correspondence has been deeply studied in the homogeneous case, among other authors, by A. Iarrobino in a long series of papers. For a modern treatment and a list of references, we refer to a book by Iarrobino and Kanev [**28**]. Every Artinian Gorenstein graded algebra $A = k[x_1, \ldots, x_n]/I$ of socle degree s, corresponds, up to scalar, to a form of degree s in a polynomial ring $P = k[y_1, \ldots, y_n]$. In particular this correspondence is bijective and it is compatible with the action of the linear group $GL_n(k)$. The study of the geometric objects arising from this correspondence between forms and ideals is a classical theme in algebraic geometry and commutative algebra. Notice that from a categorial point of view, Macaulay's correspondence is a particular case of Matlis duality.

J. Emsalem and A. Iarrobino presented the role of Macaulay's Inverse System in the study of Artinian local algebras (see [**30**] and [**18**]). We follow their approach.

Let $R = K[\![x_1, \ldots x_n]\!]$ be the ring of the formal power series with maximal ideal $\mathfrak{n} = (x_1, \ldots, x_n)$ and let $P = K[y_1, \ldots, y_n]$ be a polynomial ring. P has a structure of R-module by means of the following action

$$\begin{array}{rccc} \circ: & R \times P & \longrightarrow & P \\ & (f, g) & \to & f \circ g = f(\partial_{y_1}, \ldots, \partial_{y_n})(g) \end{array}$$

where ∂_{y_i} denotes the partial derivative with respect to y_i. This action can be defined bilinearly from that for monomials. If we denote by $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ then

$$x^\alpha \circ y^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!}\, y^{\beta-\alpha} & \text{if } \beta_i \geq \alpha_i \text{ for } i = 1, \cdots, n \\ \\ 0 & \text{otherwise} \end{cases}$$

$\frac{\beta!}{(\beta-\alpha)!} = \prod_{i=1}^n \frac{\beta_i!}{(\beta_i-\alpha_i)!}$.

We remark that for every $f, h \in R$ and $g \in P$, $(fh) \circ g = f \circ (h \circ g)$. Let now s be a positive integer and denote by $P_{\leq s}$ the set of polynomials of degree $\leq s$. Thus $\mathfrak{n}^{s+1} \circ g = 0$ if and only if $g \in P_{\leq s}$.

Now consider the pairing induced by $\circ$:

$$\begin{array}{cccc} \langle\ ,\ \rangle : & R \times P & \longrightarrow & k \\ & (f, g) & \to & (f \circ g)(0) \end{array}$$

which will give a canonic bijection between the ideals of R and R-submodules of P.

Let $I \subset R$ be an ideal, we define

$$I^\perp := \{g \in P \mid \langle f, g\rangle = 0 \ \ \forall f \in I\}.$$

Since $(\mathfrak{n}I \circ g)(0) = 0$ if and only if $I \circ g = 0$, it follows that

$$I^\perp = \{g \in P \mid I \circ g = 0\ \}.$$

In particular $I^\perp$ is an R-submodule of P. In fact if $g \in I^\perp$, then $f \circ g \in I^\perp$ for every $f \in R$. Observe that, if $I = \mathfrak{n}^{s+1}$, then $I^\perp$ coincides with $P_{\leq s}$. Notice that $I^\perp$ is finitely generated if and only if $A = R/I$ has finite length.

Conversely, for every R-submodule M of P, define

$$Ann_R(M) := \{g \in R \mid \langle g, f\rangle = 0 \ \ \forall f \in M\ \}.$$

Since M is an R-submodule of P one can prove that

$$Ann_R(M) = \{g \in R \mid g \circ M = 0\ \}.$$

It is easy to see that $Ann_R(M)$ is an ideal of R. If M is cyclic, i.e. $M = \langle f\rangle_R = R \circ f$ with $f \in P$, then we will write $Ann_R(f)$. We remark that $M = \langle f\rangle_R$ is a k-vector space generated by the polynomial f and all its derivatives of every order.

Let I be an ideal of R such that $A = R/I$ has finite length. Let $\mathfrak{m} = \mathfrak{n}/I$ be the maximal ideal of A. The action $\langle\ ,\ \rangle$ induces the following isomorphism of k-vector spaces (see [**18**] Proposition 2 (a)):

$$(R/I)^* \simeq I^\perp \tag{5.1}$$

where $(\ )^*$ denotes the dual induced by $\langle\ ,\ \rangle$. Hence $\dim_k R/I (= e_0(R/I)) = \dim_k I^\perp$.

There is a one-to-one correspondence between the ideals $I \subseteq R$ such that R/I is Artinian and the finitely generated R-submodules M of P. The correspondence is defined sending I to $I^\perp$, conversely M goes to $Ann_R(M)$. For a reference, see also [**18**] Proposition 2, Corollary 2.

A more precise result can be formulated for Artinian Gorenstein rings as follows (see [**27**], Lemma 1.2.).

THEOREM 5.2. *A local ring $A = R/I$ is an Artinian Gorenstein local ring of socle degree s if and only if its dual module $I^{\perp}$ is a cyclic R-submodule of P generated by a polynomial $F \in P$ of degree s.*

In other words, the above result says that there is the following one-to-one correspondence of sets:

$$\left\{ \begin{array}{c} I \subseteq R \text{ ideal such that} \\ R/I \text{ is Artinian Gorenstein} \\ \text{with socledegree}(R/I) = s. \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} M = R \circ F \ \ R\text{-cyclic submodule of } P \\ \text{with degree } F = s. \\ \end{array} \right\}$$

$$\begin{array}{ccc} I & \longrightarrow & I^{\perp} \\ Ann_R(F) & \longleftarrow & M =< F >_R \end{array}$$

If $F, G \in P$, then $< F >_R = < G >_R$ if and only if $G = u \circ F$ for some unit u in R.

Now, let A_F denote the Artinian Gorenstein algebra associated to $F \in P$, i.e.

$$A_F = R/Ann_R(F).$$

The isomorphism (5.1) preserves the length and the Hilbert function (see [**27**] page 10). As in the graded case, it is possible to compute the Hilbert function via the Inverse System.

If we let

$$(I^{\perp})_i := \frac{I^{\perp} \cap P_{\leq i} + P_{<i}}{P_{<i}} \tag{5.2}$$

then, by (5.1), we can prove

$$HF_{R/I}(i) = \dim_k (I^{\perp})_i. \tag{5.3}$$

Let's consider the following examples.

EXAMPLE 5.3. Let $R = k[[x, y]]$ and $P = k[x, y]$. Consider the R-submodule of P generated by $F = y^6 + xy^4$ and $G = x^4 + y^3$. We have

$I = Ann_R(< F, G >_R) = Ann_R(< F >_R) \cap Ann_R(< G >_R) = (x^2y, x^5, y^5 - 30xy^3)$.

An easy computation shows that

$$< F, G >_R = < F, G, x^4, y^4, x^3, xy^2, y^3, x^2, xy, y^2, x, y, 1 >_k$$

as k-vector space. Hence by (5.3) we get

$$HF_{R/I}(z) = 1 + 2z + 3z^2 + 3z^3 + 2z^4 + z^5 + z^6.$$

We remark that A is a Cohen-Macaulay local ring of type 2 and the generators of its socle are in degrees 4 and 6, respectively the degrees of F and G.

EXAMPLE 5.4. Let $I = (xy, y^2 - x^3) \subseteq R = k[[x, y]]$. It is easy to see that

$$I^{\perp} = < x^3 + 3y^2 >_R$$

and $< x^3 + 3y^2 >_R = < x^3 + 3y^2, x^2, x, y, 1 >_k$ as k-vector space. Hence

$$HF_{R/I}(z) = 1 + 2z + z^2 + z^3$$

We remark that the Hilbert function of $A = R/I$ is not symmetric even if A is Gorenstein.

From now on we assume that A is an Artinian Gorenstein local k-algebra where k is algebraically closed of characteristic zero. In [**27**] A. Iarrobino studied a special filtration of $G = gr_{\mathfrak{m}}(A)$ consisting of a descending sequence of ideals:

$$G = C(0) \supseteq C(1) \supseteq \cdots \supseteq C(s-2)$$

whose successive quotients

$$Q(a) = C(a)/C(a+1),$$

$a = 0, \dots, s-2$, are reflexive G-modules (see [**27**], Theorem 1.5). In particular $Q(a)$ has symmetric Hilbert function $HF_{Q(a)}$ with offset center $\frac{s-a}{2}$ (see also Proposition 1.9 [**27**]). The reflexivity of $Q(a)$ as k-vector space is induced from the nonsingular pairing on A where the ideals $(0 : \mathfrak{m}^i)$ of A correspond to $(\mathfrak{m}^i)^{\perp}$. In the graded case $0 : \mathfrak{m}^i = \mathfrak{m}^{s+1-i}$, hence the Hilbert function of A is symmetric. When A is not graded, $(0 : \mathfrak{m}^i) \neq \mathfrak{m}^{s+1-i}$ but the duality still gives some information.

The homogeneous ideals $C(a)$ of G are defined piecewise as

$$C(a)_i = \frac{0 : \mathfrak{m}^{s+1-a-i} \cap \mathfrak{m}^i}{0 : \mathfrak{m}^{s+1-a-i} \cap \mathfrak{m}^{i+1}}$$

Theorem 5.5. *Let $(A, \mathfrak{m})$ be an Artinian Gorenstein local ring with socle degree $s \geq 2$. With the above notation, we have:*

(1) $HF_{Q(a)}(i) = HF_{Q(a)}(s-a-i)$ *for every* $i \geq 0$

(2) $HF_A(i) = \sum_{a=0}^{s-2} HF_{Q(a)}(i)$ *for every* $i \geq 0$.

The decomposition of the Hilbert function in symmetric functions $HF_{Q(a)}$ will be called Q-decomposition.

Notice that $Q(0) = G/C(1)$ is the unique (up to isomorphism) homogeneous quotient of G which is Gorenstein with the same socle degree s.

It is known that if $HF_A(n)$ is symmetric, then $G = Q(0)$ and G is Gorenstein. Hence

$$\text{G is Gorenstein} \iff HF_A(n) \text{ is symmetric} \iff G = Q(0).$$

(see [**27**], Proposition 1.7 and [**18**], Proposition 7).

The G-module $Q(0)$ plays a crucial role and it can be computed in terms of the corresponding polynomial in the inverse system. Let $F \in P$ be a polynomial of degree s and denote by F_s the form of highest degree in F, that is $F = F_s + \dots$ terms of lower degree, then

$$Q(0) \simeq R/Ann_R(F_s).$$

Example 5.6. We consider $I = (x^4, x^3 - y^2) \subseteq R = K[[x, y]]$. In this case $I^{\perp} =< y^3 + x^3 y >_R$ and

$$HS_{R/I}(z) = 1 + 2z + 2z^2 + 2z^3 + z^4$$

which is symmetric, hence $gr_{\mathfrak{m}}(A)$ is Gorenstein. Indeed in this case the associated graded ring is $gr_{\mathfrak{m}}(A) = P/(x^4, y^2)$. Notice that $A = R/I$ is not canonically graded, that is $A \not\simeq gr_{\mathfrak{m}}(A)$ as k-algebras.

The following example shows that the Q-decomposition is not uniquely determined by the Hilbert function.

EXAMPLE 5.7. Consider the numerical function

$$H = \{1, 3, 3, 2, 1, 1\}.$$

It is admissible for an Artinian Gorenstein algebra. In this case $s = 5$ and it admits the following Q-decompositions, $a = 0, 1, 2, 3$:

H	1	3	3	2	1	1
$Q(0)$	1	1	1	1	1	1
$Q(1)$		1	1	1		
$Q(2)$		1	1			

H	1	3	3	2	1	1
$Q(0)$	1	1	1	1	1	1
$Q(1)$		1	2	1		
$Q(2)$		0	0			
$Q(3)$		1				

Both can be realized. Let $R = K[[x, y, z]]$ and consider the following Artinian Gorenstein rings:

(1) $A = R/I$ where $I = Ann_R(x^5 + x^3z + x^2y^2 + y^4 + z^3)$

(2) $B = R/J$ where $J = Ann_R(x^5 + x^2y^2 + xy^3 + 2y^4 + zx^2 + z^2)$

Then $H = HF_A = HF_B$ and they correspond respectively to the above different Q-decompositions, in particular we deduce that A and B are not isomorphic.

In codimension two the Hilbert function determines the Q-decomposition. Since in codimension two being Gorenstein is equivalent being a complete intersection, Iarrobino in [**27**] asked the following question.

PROBLEM 5.8. *Can there be more than one Q-decomposition for the Hilbert function of a complete intersection?*

In this last part of the survey, we shall use the Macaulay's correspondence in problems of classification.

It is reasonable to think what should be convenient to classify cyclic submodules of P instead of Artinian Gorenstein algebras A. Emsalem studied this problem in [**18**], Section C.

Given I and J ideals of $R = k[[x_1, \dots, x_n]]$, there exists a k-algebras isomorphism

$$\phi : R/I \to R/J$$

if and only if ϕ comes from an automorphism of R sending I to J. We recall that the automorphisms of R act as replacement of x_i by $z_i = \phi(x_i)$, $i = 1, \cdots, n$, such that $\mathfrak{n} = (x_1, \dots, x_n) = (z_1, \dots, z_n)$ (see [**17**]). We encode the isomorphism ϕ by the list $x_i \longrightarrow z_i$, $i = 1, \cdots, n$.

The automorphisms of R induce univocally the automorphisms of $R/\mathfrak{n}^{s+1}$ which are special homomorphisms of k-vector spaces of finite dimension. If we assume $\mathfrak{n}^{s+1} \subset I, J$, we can say that there exists a k-algebras isomorphism

$$\phi : R/I \to R/J$$

if and only if ϕ comes from an automorphism of $R/\mathfrak{n}^{s+1}$ sending $I/\mathfrak{n}^{s+1}$ to $J/\mathfrak{n}^{s+1}$.

Passing to the dual as K-vector spaces induced by $\langle\ ,\ \rangle$,

$$Hom(\phi) = \phi^* : (R/J)^* \to (R/I)^*$$

is an isomorphism of the k-vector spaces where $(R/I)^* = I^\perp$ and $(R/J)^* = J^\perp$. Hence, in terms of the corresponding polynomials $F, G \in P$, we can prove that

$$(5.4) \qquad \phi(A_F) = A_G \quad \text{if and only if} \quad (\phi^*)^{-1}(< F >_R) =< G >_R .$$

Then $\phi(A_F) = A_G$ if and only if $(\phi^*)^{-1}(F) = u \circ G$ where u is a unit in R.

We want to use this approach for the classification of the Artinian Gorenstein local algebras of low socle degree. Assume k is algebraically closed. It is easy to classify, up to isomorphism, the Artinian Gorenstein local rings of socle degree two and embedding dimension n. In fact, using the techniques of this section, the problem can be easily rephrased in terms of the classification of the hypersurfaces of degree two in $\mathbf{P^{n-1}}$. In particular we can prove that $A \simeq A_F$ with $F = y_1^2 + \cdots + y_n^2 \in P = k[y_1, \ldots, y_n]$.

We have thus the following result.

Proposition 5.9. *An Artinian local k–algebra A of embedding dimension n is Gorenstein with socle degree two if and only if $A \cong R/I$ where*

$$I = (x_i x_j, x_n^2 - x_1^2)_{1 \le i < j \le n}.$$

Hence if $\mathfrak{m}^3 = 0$, then $A \simeq gr_{\mathfrak{m}}(A)$. It is very rare that a local ring is isomorphic to its associated graded ring. Nevertheless we will see that this happens even if $\mathfrak{m}^4 = 0$ and the Hilbert function is symmetric.

If $A \simeq gr_{\mathfrak{m}}(A)$, accordingly with Emsalem in [**18**], we say that A is *canonically graded*. The following result is surprising.

Theorem 5.10. *([**14**] Theorem 3.3.) Let $A = R/I$ be an Artinian Gorenstein local k-algebra with Hilbert function $\{1, n, n, 1\}$. Then A is canonically graded.*

Corollary 5.11. *The classification of Artinian Gorenstein local k-algebras with Hilbert function $HF_A = \{1, n, n, 1\}$ is equivalent to the projective classification of the hypersurfaces $V(F) \subset \mathbb{P}_K^{n-1}$ where F is a degree three non degenerate form in n variables.*

If $n = 1$ then it is clear that $A \cong K[[x]]/(x^4)$, so there is a only one analytic model. By using Theorem 5.10, here we present the classification in the case $n = 2$.

Proposition 5.12. *Let A be an Artinian Gorenstein local k-algebra with Hilbert function $HF_A = \{1, 2, 2, 1\}$. Then A is isomorphic to one and only one of the following algebras*

Model $A = R/I$	*Inverse system F*	*Geometry of $C = V(F) \subset \mathbb{P}_K^1$*
(x_1^3, x_2^2)	$y_1^2 y_2$	*Double point plus a simple point*
$(x_1 x_2, x_1^3 - x_2^3)$	$y_1^3 - y_2^3$	*Three different points*

Proof. By the previous theorem, we have $A \simeq gr_{\mathfrak{m}}(A) = K[y_1, y_2]/Ann(F)$ where $F \in K[y_1, y_2]$ is a degree three form in the variables y_1, y_2. Since K is an algebraic closed field, F can be decomposed as product of three linear forms L_1, L_2, L_3, i.e. $F = L_1 L_2 L_3$. We set $d = dim_K \langle L_1, L_2, L_3 \rangle$, so we only have to consider three cases. If $d = 1$, then we can assume $F = y_1^3$, but this has not the right Hilbert function.

If $d = 2$, then we can assume $F = y_1^2 y_2$. It is easy to see that $Ann(\langle y_1^2 y_2 \rangle) = (x_1^3, x_2^2)$. If $d = 3$ then we can assume $F = y_1^3 - y_2^3$. In this case we get $Ann(\langle y_1^3 - y_2^3 \rangle) = (x_1 x_2, x_1^3 - x_2^3)$. Since $V(y_1^2 y_2)$ (resp. $V(y_1^3 - y_2^3)$) is a degree three subscheme of $\mathbb{P}^1_K$ with two (resp. three) point basis we get that the algebras of the statement are not isomorphic. □

The classification of Artinian Gorenstein local rings with Hilbert function $HF_A = \{1, 3, 3, 1\}$ had been studied by Casnati and Notari in [**7**] and by Cartwright et al. in [**6**]. In [**14**] J. Elias and the author present a different proof by using Theorem 5.10 and the geometric models of the varieties defined by them are described.

Gorenstein local rings with symmetric Hilbert function are not necessarily canonically graded. The following exercise shows that we cannot extend the main result of this section to higher socle degrees.

Exercise 5.13. Let A be an Artinian Gorenstein local local k-algebra with Hilbert function $HF_A = \{1, 2, 2, 2, 1\}$. Then A is isomorphic to one and only one of the following rings:

(a) R/I with $I = (x^4, y^2)$ and $I^\perp = \langle x^3 y \rangle$ and A is isomorphic to its associated graded ring,
(b) R/I with $I = (x^4, -x^3 + y^2)$ and $I^\perp = \langle x^3 y + y^3 \rangle$. The associated graded ring is of type (a) and it is not isomorphic to R/I,
(c) R/I with $I = (x^2 + y^2, y^4)$ and $I^\perp = \langle xy(x^2 - y^2) \rangle$ and A is isomorphic to its associated graded ring.

The computation can be performed by using the techniques of this section, a different approach can be found in [**15**].

Next we consider the Artinian Gorenstein local rings A of socle degree three. Then by using Theorem 5.5, we know that

$$HF_A = \{1, m, n, 1\}$$

with $m \geq n$. J. Elias and M.E. Rossi recently proved the following result (see [**14**], Theorem 4.1.).

Theorem 5.14. *There exists an isomorphism between the Artinian Gorenstein local k-algebras $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ with Hilbert function $\{1, m, n, 1\}$, $m \geq n$ if and only if $Q_A(0) \simeq Q_B(0)$ as graded k-algebras.*

Since $Q_A(0)$ is Gorenstein with Hilbert function $\{1, n, n, 1\}$, we obtain the following result which reduces the problem of the classification of the Artinian Gorenstein algebras of socle degree three to the classification of the projective cubics.

Corollary 5.15. *The classification of Artinian Gorenstein local k-algebras with Hilbert function $\{1, m, n, 1\}$, $m \geq n$, is equivalent to the projective classification of cubic hypersurfaces $H = V(F) \subset \mathbb{P}^{n-1}_k$.*

The crucial point in the proof of Theorem 5.14 is the following. Let A be an Artinian Gorenstein local k–algebra with Hilbert function $\{1, m, n, 1\}$, then $A \cong R/Ann_R(F)$ where

$$F = F_3 + y_{n+1}^2 + \cdots + y_m^2 \in P$$

with F_3 a non-degenerate degree three form in $P_n = k[y_1, \ldots, y_n]$ ($F = F_3$ if $n = m$). Non-degenerate in $P_n = k[y_1, \ldots, y_n]$ means that $F_3 \in P_n$ and $R/Ann_R(F_3)$ has embedding dimension n.

Starting from the above result we can prove a structure theorem for Artinian Gorenstein local k–algebra $A = R/I$ with maximal ideal $\mathfrak{m}$, embedding dimension m and socle degree three. Denote by $R_n := k[[x_1, \ldots, x_n]]$ with $n \leq m$ the subring of $R = k[[x_1, \ldots, x_m]]$.

Theorem 5.16. *Let A be an Artinian local k–algebra of embedding dimension m and let $n = HF_A(2)$.*

A is Gorenstein of socle degree three if and only if $n \leq m$ and there exists a non-degenerate cubic form $F_3 \in P_n$ such that $A \cong R/I$ where

$$I = \begin{cases} Ann_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i<j,\, n+1 \leq j \leq m} & \text{if } n < m \\ \\ Ann_R(F_3)R & \text{if } n = m \end{cases}$$

being $\sigma \in R_n$ any form of degree 3 such that $\sigma \circ F_3 = 1$.

For a proof we refer to [**9**].

Taking advantage of the classification of the cubics in $\mathbf{P^2}$, from the above results we may deduce the classification, up to isomorphism, of the Artinian Gorenstein local k-algebras of socle degree and embedding dimension ≤ 3 (see [**14**]).

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