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Hilbert functions and free resolutions of filtered modules

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The material of this presentation is essentially part of the papers [38], [34], jointly with G. Valla, University of Genoa (Italy), and [39], [40] jointly with L. Sharifan, University of Teheran (Iran). We refer to these papers for more details and complete proofs.

We present here several examples, all performed using CoCoA [5].

1 Results, examples and problems on the Hilbert functions of local rings

We start the presentation with an overview on some classical problems on the Hilbert Functions in order to guide the reader in a possible route through this area of dynamic mathematical activity. First we present a list of problems in the specific case of the $\mathfrak{m}$-adic filtration on a Cohen-Macaulay local ring because also in this case too many interesting questions are still open. Later we will extend the definitions to the more general setting of filtrations on a module.

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field k. Denote by $\mu(\ \ )$ the minimal number of generators of an ideal of A. The Hilbert function of A is, by definition

$$HF_A(n) := dim_k\ \mathfrak{m}^n/\mathfrak{m}^{n+1} = \mu(\mathfrak{m}^n)$$

for every $n \geq 0$. Hence HF_A is the Hilbert function of the homogeneous k-standard algebra

$$gr_{\mathfrak{m}}(A) = \oplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$$

which is the associated graded ring of $\mathfrak{m}$. The graded algebra $gr_{\mathfrak{m}}(A)$ corresponds to a relevant geometric construction and it has been studied extensively. Namely, if A is the localization at the origin of the coordinate ring of an affine variety V passing through 0, then $gr_{\mathfrak{m}}(A)$ is the coordinate ring of the *tangent cone* of V, which is the cone composed of all lines that are limiting positions of secant lines to V in 0. The *Proj* of this algebra can also be seen as the *exceptional set* of the *blowing-up* of V in 0.

The problems that I shall present here can be rephrased and extended to a related graded k-standard algebra defined starting from any ideal I of A, the Fiber Cone of I

$$F_{\mathfrak{m}}(I) := \oplus_{n\geq 0}I^n/\mathfrak{m}I^n$$

whose Hilbert function $HF_{F_{\mathfrak{m}}(I)}(n) = \mu(I^n)$ controls the number of generators of the powers of I.

By the Hilbert-Serre Theorem, the generating function of HF_A is a power series $HS_A(z) = \sum_{n\geq 0} HF_A(n)z^n$ which can be written as

$$HS_A(z) = \frac{h_A(z)}{(1-z)^d}$$

where $h_A(z)$ is a polynomial with integer coefficients such that $h_A(1) \neq 0$ and d is the Krull dimension of A. For $n \gg 0$ $HF_A(n)$ agrees with a polynomial $HP_A(X)$ which has rational coefficients and degree $d-1$ and it is called the Hilbert polynomial of A. We can write

$$HP_A(X) = \sum_{i=0}^{d-1} (-1)^i e_i(\mathfrak{m}) \binom{X+d-i-1}{d-i-1}.$$

The integers $e_0(\mathfrak{m}), e_1(\mathfrak{m}), \dots, e_{d-1}(\mathfrak{m})$ are uniquely determined by $\mathfrak{m}$ and are known as the Hilbert coefficients. In particular e_0 is the multiplicity of A. Classically a related polynomial has been introduced, the Hilbert-Samuel polynomial, that is $\lambda(A/\mathfrak{m}^{n+1})$ for $n \gg 0$. It is denoted by $HP_A^1(X)$ and

$$HP_A^1(X) = \sum_{i=0}^{d} (-1)^i e_i(\mathfrak{m}) \binom{X+d-i}{d-i}. \tag{1}$$

If there is not confusion, we simply write e_i instead of $e_i(\mathfrak{m})$. We can prove that for every $i \geq 0$

$$e_i = \frac{h_A^{(i)}(1)}{i!}$$

where $0! = 1$ and $h_A^{(0)}(1) = h_A(1)$.

In the case of a k-standard graded algebra, the Hilbert function is well understood, at least in the Cohen-Macaulay case. Instead, very little is known in the local case.

A result by Srinivas and Trivedi (see [47] and also [41]) says that the number of Hilbert functions of a Cohen-Macaulay local rings with fixed dimension and multiplicity is finite, but the problem concerning the characterization of the numerical functions which are admissible for a Cohen-Macaulay local ring is widely open.

Due to this lack of information, a long list of papers have been written on the problem to find constraints on the possible Hilbert functions of a Cohen-Macaulay local ring.

Assume $(A, \mathfrak{m})$ be Cohen-Macaulay local ring of dimension one of given embedding dimension $(= \dim_k \mathfrak{m}/\mathfrak{m}^2)$. J. Elias (see [9]) characterized the Hilbert-Samuel polynomials $HP_A^1(X) = e_0(X+1) - e_1$ of A. It is clear that this result gives some information on the Hilbert functions. For example it says that $HS(z) = \frac{1+z-z^2}{1-z}$ is an admissible Hilbert series for a k-standard algebra of dimension one, but it not the Hilbert series of a 1-dimensional Cohen-Macaulay local ring.

But the following problem is still open, even if we consider $\dim A = 1$.

PROBLEM 1. Characterize the possible Hilbert functions of a Cohen-Macaulay local ring of dimension one.

The question has a clear geometric meaning related to the singularities of the affine curves which are arithmetically Cohen-Macaulay. From the algebraic point of view, the problem mainly comes from the fact that the Cohen-Macaulyness of A (resp. Gorenstein, domain, ...) does not imply $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay (resp. Gorenstein, domain, ...).

Example 1.1. Consider the power series $A = k[[t^4, t^5, t^{11}]]$. This is a one-dimensional local domain and its associated graded ring is

$$gr_{\mathfrak{m}}(A) = k[x, y, z]/(xz, yz, z^2, y^4)$$

which is not Cohen-Macaulay and the Hilbert series of A is

$$HS_A(z) = \frac{1+2z+z^3}{1-z}$$

In fact $A = k[[x,y,z]]/I$ where $I = (x^4 - yz, y^3 - xz, z^2 - x^3y^2)$ and hence $gr_{\mathfrak{m}}(A) = k[x,y,z]/I^*$ where $I^* = (xz, yz, z^2, y^4)$ is the ideal generated by the initial forms of the elements of I. In this case dim$A$ = dim $gr_{\mathfrak{m}}(A) = 1$, but depth $gr_{\mathfrak{m}}(A) = 0$.

Here if $a \in A$ is a non-zero element and n is the greatest integer such that $a \in \mathfrak{m}^n$ we let

$$a^* := \overline{a} \in \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

and call it *the initial form* of a in $\mathrm{gr}_{\mathfrak{m}}(A)$.

The problem is still open if we concentrate our interest to the complete intersections.

Example 1.2. Consider the coordinate ring A of the monomial curve parametrized by (t^6, t^7, t^{15}). The 1-dimensional local domain A is a complete intersection. In fact $A = k[[x,y,z]]/I$ where $I = (y^3 - xz, x^5 - z^2)$. Since $I^* = (xz, z^2, y^3z, y^6)$, one has

$$HS_A(z) = \frac{1+2z+z^2+z^3+z^5}{1-z}.$$

In this case the associated graded ring is not Cohen-Macaulay. Notice that in the graded case the Hilbert function of a complete intersection is always symmetric, it is no longer true in the local case. The point is that even if A is Gorenstein, the associated graded ring can lose this property.

We say that the *Hilbert function of A is not decreasing* if

$$HF_A(n+1) \geq HF_A(n)$$

for every n. Obviously the property is verified if $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay, but this is not a necessary requirement (see Examples 1.1 and 1.2). Unfortunately in the Cohen-Macaulay local case it can happen that $HF_A(2) = \mu(\mathfrak{m}^2) < HF_A(1) = \mu(\mathfrak{m})$. The first 1-dimensional examples were given by Herzog and Waldi in 1975, by Eakin and Sataye in 1976. As far as I know, Molinelli and Tamone in 1999 gave the following example which has multiplicity and embedding dimension smaller than the previous examples. It is enough to consider

$$A = k[[t^{13}, t^{19}, t^{24}, t^{44}, t^{49}, t^{54}, t^{55}, t^{59}, t^{60}, t^{66}]].$$

In this case $HF_A(2) = 9 < HF_A(1) = 10$. It would be interesting to notice that the Hilbert function is not decreasing if the multiplicity is smaller than $\mu(\mathfrak{m}) - d + 4$ and, in this case, the multiplicity of A is $\mu(\mathfrak{m}) - d + 4 = 13$. In 1980 Orecchia proved that, for all $b \geq 5$, there exists a reduced one-dimensional local ring of embedding dimension b and decreasing Hilbert function. Finally P. Roberts in 1982 built ordinary singularities with decreasing Hilbert function and embedding dimension at least 7.

J. Elias (see [10]) gave a positive answer to a problem stated by J. Sally by proving that the Hilbert function of a Cohen-Macaulay local ring of dimension one and embedding dimension at most three is not decreasing.

Several computations leads us to think that these pathologies do not appear if the local ring is Gorenstein. Then we ask the following question:

PROBLEM 2. Is the Hilbert function of a Gorenstein local ring of dimension one not decreasing?

Very partial results have been proved in the case of a complete intersection. T. Puthenpurakal gave a positive answer if $A = k[[x,y,z,w]]/I$ where I is generated by a regular sequence of height at most three.

Interesting problems also come from the Artinian case, under the assumption that A is Gorenstein or, more in general, level.

Iarrobino (see [25]) proved a necessary (not sufficient) constraint on a numerical function for being the Hilbert function of a **Gorenstein Artinian local ring**, but a complete characterization for this class of rings is still an open problem, except for the case of codimension two.

In fact if we consider an Artinian local ring $A = k[[x,y]]/I$ with Hilbert function $HF = \{(1,2,h_2,\dots,h_s)\}$, Macaulay, by using the device of the inverse system, proved that if $I = (f,g)$ is a complete intersection, then

$$|h_i - h_{i+1}| \leq 1 \text{ for all } \ i. \tag{2}$$

In 1978 S.C. Kothari answered several questions raised by Abhyankar concerning the Hilbert function of a pair of plane curves and he proved the same result by studying the complicate structure of I^*. Starting from the same point of view, recently S. Goto, W. Heinzer and M-K. Kim examined the ideal I^* of a complete intersection of height two and they reproved Macaulay's result. Several authors studied the variety parametrizing all height two ideals with a fixed Hilbert function, among others J. Briancon, A. Iarrobino, Yameogo and Gottsche. In particular J. Briancon and A. Iarrobino proved that (2) gives a complete characterization of the Hilbert functions of a Gorenstein Artinian local ring of codimension two (complete intersections) and recently V. Bertella extended the result to the Level Artinian rings.

In the last section we will present a short proof of Briancon-Iarrobino's result as a consequence of a different approach. In codimension three, we can state the following problem:

PROBLEM 3. Characterize the Hilbert functions of the Artinian local ring $A = k[[x,y,z]]/I$ where $I = (F_1,F_2,F_3)$ is generated by a regular sequence.

We will see that, for example, $\{(1,3,4,4,1,1,1)\}$ is not the Hilbert function of a Gorenstein Artinian local ring. Partial results will be presented later.

Due to the pioneering work by Northcott (see [28]) in the 60's, several efforts have been made to better understand the Hilbert function of a Cohen-Macaulay local ring, also in relation with the Hilbert coefficients (asymptotic information) and the homological invariants of the corresponding tangent cone. J. Sally carried on Northcott's work (see [42], [43]) by proving interesting results and asking as well as challenging questions. Several improvements along this line has been proved in the last years (J.Elias, C. Huneke, S.Itoh, A. Ooishi, G. Valla, W. Vasconcelos, ...), often extending the framework to the Hilbert functions associated to an $\mathfrak{m}$-primary ideal or, more in general, to a filtration.

The first Hilbert coefficient, $e_0(\mathfrak{m})$ or simply e, is the multiplicity and, due to its geometric meaning, has been studied very deeply. The other coefficients are not as well understood, either geometrically or in terms of how they are related to algebraic properties of the ideal or the ring. Northcott proved that, if $(A,\mathfrak{m})$ is a Cohen-Macaulay local ring, then

$$e_1 \geq e - 1.$$

Thus, for example, the series $\frac{1+z-z^2}{1-z}$ cannot be the Hilbert series of a Cohen-Macaulay local ring of dimension one because $e_1 = 0$, while $e = 2$. On the other hand, the above series is not admissible for a local Cohen-Macaulay local ring also because does not verify Abhyankar's bound. Abhyankar (see [1]) proved that if A is Cohen-Macaulay

$$e \geq h + 1$$

where $h = \mu(\mathfrak{m}) - d$ is the so-called embedding codimension of A ($h \geq 0$ and $h = 0$ if and only A is regular). In particular

$$e = h + 1 + \lambda(\mathfrak{m}^2/J\mathfrak{m}) \tag{3}$$

for every minimal reduction J of $\mathfrak{m}$. D.G. Northcott and J. Sally studied the minimal values of e and e_1 in terms of these invariants. We have

$$e = h + 1 \iff e_1 = e - 1 \iff HS_A = \frac{1 + hz}{(1-z)^d} \iff \mathfrak{m}^2 = J\mathfrak{m} \tag{4}$$

for every minimal reduction J of $\mathfrak{m}$. If this is the case $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay.
These results can be easily proved by using the device of the superficial elements (see Section 2). S. Goto and K. Nishida (see [15]) have been able to extend Northcott's bound to the case of an $\mathfrak{m}$-primary ideal, avoiding the assumption that the ring is Cohen-Macayulay.

On the base of this result, A. Corso (see [5]) can handle in this wild generality a stronger upper bound for e_1 proved by J. Elias and G. Valla (see [11]) in the Cohen-Macaulay case.

The integer, e_1, has been recently interpreted by C. Polini, B. Ulrich and W. Vasconcelos as a tracking number of the Rees algebra of A in the set of all such algebras with the same multiplicity. Under various circumstances, it is also called the *Chern number or coefficient* of the local ring A. An interesting list of questions and conjectural statements about the values of e_1 for filtrations associated to an $\mathfrak{m}$-primary ideal of a local ring A have been presented in a paper by W. Vasconcelos (see [51]). Relevant improvements have been recently proved by S. Goto, L. Ghezzi, J. Hong, K. Ozeky, T.T. Phuong and W. Vasconcelos (see [13]).

In 1983 J. Sally (see [43]) conjectured that

$$e = h + 2 \implies HS_A(z) = \frac{1 + hz + z^s}{(1-z)^d}$$

for some integer $2 \leq s \leq e - 1$. Sally proved it in dimension one and, for every integer s, she gave an example:

$$A_2 = k[[t^e, t^{e+1}, t^{e+3}, \dots, t^{2e-1}]]$$
$$A_s = k[[t^e, t^{e+1}, t^{e+s+1}, t^{e+s+2}, \dots, t^{2e-1}, t^{2e+3}, t^2e+4, \dots, t^{2e+s}]],\ 3 \leq s \leq e-2$$
$$A_{e-1} = k[[t^2, t^{e+1}, t^{2e+3}, t^{2e+4}, \dots, t^{3e-1}]].$$

After 13 years G. Valla and M.E. Rossi solved the conjecture (see [34]). Their proof deeply involved the use of the Rattlif-Rush filtration. H.S. Wang presented a different proof in [53] which involves a technical study of graded diagonals of the Sally module. Sally's conjecture can be proved by reducing the problem to dimension two. While the proof in the one-dimensional case is quite easy, the proof in dimension two is very complicated. Several authors generalized the result to a more general setting (S. Huckaba, A. Corso, C. Polini and M. Vaz Pinto, J. Elias, M.E. Rossi and G. Valla) but always using the devices of the original proof of G. Valla and M.E. Rossi.

If $e = h + 3$ and A is Gorenstein, J. Sally proved that

$$HS_A(z) = \frac{1 + hz + z^2 + z^3}{(1-z)^d}$$

and $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay.

G. Valla and M.E. Rossi characterized the admissible Hilbert functions for a one-dimensional Cohen-Macaulay ring with multiplicity $e = h + 3$ (see [36]). Moreover they reproved Sally's result and they showed that

$$\tau < h \implies HS_A(z) = \frac{1 + hz + z^t + z^s}{(1-z)^d} \quad \text{where } 2 \leq t \leq s \leq e - 1$$

where τ is the Cohen-Macaulay type of A. If this is the case $depth gr_{\mathfrak{m}}(A) \geq d-1$.

The following example given by H.S. Wang shows that the above result on the depth of $gr_{\mathfrak{m}}(A)$ is no longer true if τ is almost maximal, i.e. $\tau = h, h+1$ ($\tau \leq h+1$).

Example 1.3. Let $(A, \mathfrak{m})$ be the two dimensional local Cohen-Macaulay ring

$$k[\![x, y, t, u, v]\!]/(t^2, tu, tv, uv, yt - u^3, xt - v^3),$$

Let $\mathfrak{m}$ be the maximal ideal of A. One has $e = h + 3 = 6$, the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(A)$ has depth zero and

$$HS_A(z) = \frac{1 + 3z + 3z^3 - z^4}{(1-z)^2}.$$

In the previous example depth $gr_{\mathfrak{m}}(A) = d - 2 = 0$. On the analogy with the previous case ($e = h + 2$), it is natural the following problem:

PROBLEM 4. Let A be a Cohen-Macaulay local ring with $e = h + 3$. Is depth $gr_{\mathfrak{m}}(A) \geq d - 2$ **?**

In the classical case of a Cohen-Macaulay local ring A, as far as the higher Hilbert coefficients are concerned, it is a result of M. Narita (see [27]) and J. Sally (see [44]) that

$$e_2 \geq e_1 - e + 1 \geq 0. \tag{5}$$

If $e_2 = 0$, Narita proved that $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay and, for large n, $\mathfrak{m}^n$ has reduction number one. Unfortunately if the first equality holds, we cannot deduce that the associated graded ring is Cohen-Macaulay or of almost maximal depth. In Example 1.3, A is a Cohen-Macaulay local ring of dimension 2, $e_2 = e_1 - e + 1$, but depth $gr_{\mathfrak{m}}(A) = 0$. In general, it seems that the the normality of the ideal $\mathfrak{m}$ yields non trivial consequences on the Hilbert coefficients of A and, ultimately, on depth $\mathrm{gr}_{\mathfrak{m}}(A)$ and the Hilbert function A. Corso, C. Polini and M.E. Rossi in [6] proved that

$$\mathfrak{m} \text{ normal}, \ \ e_2 = e_1 - e + 1 \quad \Longrightarrow HS_A(z) = \frac{1 + hz + (e-h-1)z^2}{(1-z)^d} \text{ and } \ \mathfrak{m}^3 = J\mathfrak{m}^2$$

for every minimal reduction J of $\mathfrak{m}$ and hence $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay. This result gave a positive answer to a question raised by G. Valla in [50]. The key point is a theorem by Itoh on the normalized Hilbert coefficients of ideals generated by a system of parameters.

On the analogy of the extension of Northcott's result (due to S. Goto and K. Nishida), it would be interesting to introduce a correction term for e_2 in order to extend (5) to any local ring, not necessarily Cohen-Macaulay.

Unfortunately, the positivity of the Hilbert coefficients stops with e_2. Indeed, M. Narita showed that e_3 can be negative (see also Example 1.3). However, a remarkable result of S. Itoh says that $e_3 \geq 0$ provided $\mathfrak{m}$ is normal (see [26]). If the equality holds, then $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay. A nice proof of this result was also given by S. Huckaba and C. Huneke. We do not know if Itoh's result holds if we consider the $\mathfrak{q}$-adic filtration where $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal. As far I know there are not negative answers to the following natural question.

PROBLEM 5. Does $\mathfrak{m}$ normal imply $e_i \geq 0$ for every i ?

The previous problem is related to the asymptotic behavior of the associated graded ring of the powers of the maximal ideal and it has some relation with the Grauert-Riemenschneider vanishing theorem.

Another relevant interest goes toward the study of the reduction number. As we can realize in (3) and (4), there is a relationship between Hilbert functions and minimal reductions. Let J be a minimal reduction of $\mathfrak{m}$, we recall that the reduction number w.r.t. J is defined by

$$r_J(\mathfrak{m}) := \min\{r \in \mathbf{N} \quad : \quad \mathfrak{m}^{r+1} = J\mathfrak{m}^r\}$$

We can easily prove that if $r_J(\mathfrak{m}) \leq 2$ for some minimal reduction J, then the Hilbert function is known

$$HS_A(z) = \frac{1 + hz + (e-h-1)z^2}{(1-z)^d}.$$

Also the converse holds, that is the shape of the previous Hilbert series forces $r_J(\mathfrak{m}) \leq 2$ for every J and hence $gr_{\mathfrak{m}}(A)$ to be Cohen-Macaulay. This had been proved by J. Elias and G. Valla in [11].

If dim $A = 1$, or more in general if depth $gr_{\mathfrak{m}}(A) \geq d-1$, then $r_J(\mathfrak{m}) \leq e-1$ for every minimal reduction J.

If dim $A \leq 2$, or more in general if depth $gr_{\mathfrak{m}}(A) \geq d-2$, M.E. Rossi in [35] proved that

$$r_J(\mathfrak{m}) \leq e_1 - e + 2$$

It is natural to wonder if, given a Cohen-Macaulay local ring $(A, \mathfrak{m})$ of dimension d, does exist a linear function $f(e_0, e_1, \dots, e_{d-1}, d)$ and a minimal reduction J such that

$$r_J(\mathfrak{m}) \leq f(e_0, e_1, \dots, e_{d-1}, d).$$

The study of the reduction number w.r.t. a minimal reduction J, led W. Vasconcelos to enlarge the list of the blowup algebras by introducing the Sally module $S_J(\mathfrak{m}) = \oplus \mathfrak{m}^{n+1}/J^n\mathfrak{m}$. Some results on the Sally module will be presented in Section 3.

2 Basic facts and bounds on the first Hilbert coefficient of filtered modules

We remark that the graded algebra $gr_{\mathfrak{m}}(A)$ can also be seen as the graded algebra associated to a filtration of the ring itself, namely the $\mathfrak{m}$-adic filtration $\{\mathfrak{m}^j\}_{j\geq 0}$. This gives an indication of a possible natural extension of the theory to general filtrations of a finite module over the local ring $(A, \mathfrak{m})$. The extension of the classical theory to general filtrations of a module, besides an intrinsic interest, has one more relevant motivation. In the next section we will show as the theory of the Hilbert function of filtered modules provides results on blowing-up algebras, as the Fiber Cone and the Sally module, without further efforts.

We shall present some basic facts, in particular the notion of **superficial** element will be a fundamental tool in our approach. The definition was given in Zariski- Samuel, pg.285 where it is shown how to use this concept for devising proofs by induction on the dimension. We will care only of the purely algebraic meaning of this notion, even if superficial elements play a relevant role also in Singularity theory, as shown by Bondil and Le Dung Trang (see [2]).

We will show how this approach works in proving suitable extensions of two classical bounds on the first Hilbert coefficient: the lower bound by Northcott and an upper bound by Huckaba, see [28], [21]. We refer to [38] for further results on Hilbert coefficients, always by using this approach.

Let A be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated A-module. Let $\mathfrak{q}$ be an ideal of A; a $\mathfrak{q}$-filtration $\mathbb{M}$ of M is a collection of submodules M_j such that

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_j \supseteq \cdots.$$

with the property that $\mathfrak{q}M_j \subseteq M_{j+1}$ for $j \geq 0$. We consider a *good* $\mathfrak{q}$*-filtration* of M. This means that $M_{j+1} = \mathfrak{q}M_j$ for all large j. A good $\mathfrak{q}$-filtration is also called a *stable* $\mathfrak{q}$*-filtration.*
For example, the $\mathfrak{q}-$adic filtration on M defined by $M_j := \mathfrak{q}^j M$ is clearly a good $\mathfrak{q}$-filtration.

Several problems we have discussed in Section 1. can be properly rephrased in this general setting, replacing A by M, the $\mathfrak{m}$-adic filtration $\{\mathfrak{m}^j\}$ by $\{M_j\}$ a good $\mathfrak{q}$-filtration on M.

If N is a submodule of M, it is clear that $\{(N + M_j)/N\}_{j\geq 0}$ is a good $\mathfrak{q}$-filtration of M/N which we denote by $\mathbb{M}/N$.

Given the good $\mathfrak{q}$-filtration $\mathbb{M}$ on M we let $gr_{\mathfrak{q}}(A) := \oplus_{j\geq 0}\mathfrak{q}^j/\mathfrak{q}^{j+1}$, $gr_{\mathbb{M}}(M) := \oplus_{j\geq 0}M_j/M_{j+1}$. We know that $gr_{\mathbb{M}}(M)$ is a graded $gr_{\mathfrak{q}}(A)$-module; further each element $a \in A$ has a natural image a^* in $gr_{\mathfrak{q}}(A)$ which is 0 if $a = 0$, is $a^* = \overline{a} \in \mathfrak{q}^t/\mathfrak{q}^{t+1}$ if $a \in \mathfrak{q}^t \setminus \mathfrak{q}^{t+1}$.

From now on we shall require the assumption that the length of $M/\mathfrak{q}M$, which we denote by $\lambda(M/\mathfrak{q}M)$, is finite. In this case there exists an integer s such that $\mathfrak{m}^s M \subseteq (\mathfrak{q} + (0 :_A M))M$, hence the ideal $\mathfrak{q} + (0 :_A M)$ is primary for the maximal ideal $\mathfrak{m}$. Also the length of M/M_j is finite for all $j \geq 0$. In this setting the Hilbert function of the filtration $\mathbb{M}$ is the numerical function

$$HF_{\mathbb{M}}(j) := \lambda(M_j/M_{j+1}).$$

As in Section 1. we denote by $e_i(\mathbb{M})$ the **Hilbert coefficients** of $\mathbb{M}$ coming from the corresponding Hilbert polynomial $HP_{\mathbb{M}}(X)$. We denote by $HS_{\mathbb{M}}(z)$ the Hilbert series.

The Hilbert coefficient e_0 is the multiplicity of $\mathbb{M}$; we know that (see Bruns-Herzog's book, Proposition 11.4)

$$e_0(\mathbb{M}) = e_0(\mathbb{N}) \tag{6}$$

for every couple of good $\mathfrak{q}$-filtrations. Also, if M is Artinian, then $e_0(\mathbb{M}) = \lambda(M)$.
If we consider the classical $\mathfrak{q}$-adic filtration on A, we will denote the Hilbert coefficients by $e_i(\mathfrak{q})$.

We recall the main tools we will use later:

• *Valabrega-Valla criterion*: given the ideal $I = (a_1, \ldots, a_r)$ in A with $a_i \in \mathfrak{q} \setminus \mathfrak{q}^2$, the elements $a_1^*, \ldots, a_r^*$ form a regular sequence on $gr_{\mathbb{M}}(M)$ if and only if they form a regular sequence on M and $IM \cap M_j = IM_{j-1}$ for every $j \geq 1$.

• $\mathbb{M}$*-superficial elements*: an element $a \in \mathfrak{q}$ is called $\mathbb{M}$-superficial for $\mathfrak{q}$ if there exists a non-negative integer c such that

$$(M_{j+1} :_M a) \cap M_c = M_j$$

for every $j \geq c$. If we assume that M has positive dimension, then every superficial element a for $\mathfrak{q}$ has order one, that is $a \in \mathfrak{q} \setminus \mathfrak{q}^2$. Further, since we are assuming that the residue field $A/\mathfrak{m}$ is infinite, it is well known that superficial elements do exist. Moreover, if q is $\mathfrak{m}$-primary, then any superficial element $a \in \mathfrak{q} \setminus \mathfrak{q}\mathfrak{m}$, hence it is part of a minimal system of generators of $\mathfrak{q}$.

The obstacle to finding superficial sequences is that one needs to check the previous relations for all large n. We can circumvent this by using the equivalent notion of filter-regular sequence in $gr_{\mathbb{M}}(M)$. In particular an element $a \in \mathfrak{q}$ is $\mathbb{M}$-superficial for $\mathfrak{q}$ if and only if $a \in \mathfrak{q} \setminus \mathfrak{q}^2$ and a^* is a filter regular element on $gr_{\mathbb{M}}(M)$.

We present useful properties of the superficial elements. In the following let a be an $\mathbb{M}$-superficial element for $\mathfrak{q}$ and let d be the dimension of M. We have

a) $\dim(M/aM) = d - 1$

b) a M-regular $\iff$ depth $M > 0$

c) $j \geq 1$, depth $M/aM \geq j \iff$ depth $M \geq j + 1$

d) a^* $gr_{\mathbb{M}}(M)$-regular $\iff$ depth $gr_{\mathbb{M}}(M) > 0$

e) *Sally's machine*: depth $gr_{\mathbb{M}/aM}(M/aM) \geq 1 \iff$ depth $gr_{\mathbb{M}}(M) \geq 2$

f) $e_j(\mathbb{M}) = e_j(\mathbb{M}/aM)$ for every $j = 0, \dots, d-2$

g) $e_{d-1}(\mathbb{M}/aM) = e_{d-1}(\mathbb{M}) + (-1)^{d-1}\lambda(0 :_M a)$

h) a^* is a regular element on $gr_{\mathbb{M}}(M)$ if and only if $HS_{\mathbb{M}}(z) = \frac{HS_{\mathbb{M}/aM}(z)}{1-z}$ if and only if a is M-regular and $e_d(\mathbb{M}) = e_d(\mathbb{M}/aM)$

Assume depth $M > 0$, then $a \in \mathfrak{q}$ is $\mathbb{M}$-superficial for $\mathfrak{q}$ if and only if $e_j(\mathbb{M}) = e_j(\mathbb{M}/aM)$ for every $j = 0, \dots, d-1$.

The above properties can be easily proved by the definition of superficial element and by the following result. It is the so called **Singh's formula** because the corresponding equality in the classical case was obtained by B.Singh (see [46]).

Lemma 2.1. *Let $a \in \mathfrak{q}$; then for every $j \geq 0$ we have*

$$HF_{\mathbb{M}}(j) = \sum_{n=0}^{j} HF_{\mathbb{M}/aM}(n) - \lambda(M_{j+1} : a/M_j).$$

It is a nice consequence of this module-theoretic approach to remark that if $\mathbb{M}$ and $\mathbb{N}$ are two $\mathfrak{q}$-filtrations on the same module M, then there exists an element $a \in \mathfrak{q}$ which is superficial for both.

In fact it is enough to consider the filtered module $M \oplus M$ endowed with the $\mathfrak{q}$-filtration $\mathbb{M} \oplus \mathbb{N}$ and to remark that a is superficial for this filtration if and only if a is superficial for both $\mathbb{M}$ and $\mathbb{N}$ (this remark is due to D. Conti). As a consequence we deduce that, if the residue field is infinite, we can always find an element $a \in \mathfrak{q}$ which is superficial for a finite number of $\mathfrak{q}$-filtrations on M.

Assume dim $M \geq 2$, by using f), g) and the above fact, we can easily prove that if $\mathbb{M}$ and $\mathbb{N}$ are two $\mathfrak{q}$-filtrations on the same module M, there exists an element $a \in \mathfrak{q}$ superficial for $\mathbb{M}$ and $\mathbb{N}$ such that:

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) = e_1(\mathbb{M}/aM) - e_1(\mathbb{N}/aM) \tag{7}$$

The above property will be useful for proving results on e_1 by induction on the dimension of M.

Assume $\dim M = d > 0$ and denote by $H^0_{\mathfrak{m}}(M)$ the $0-$th local cohomology with respect to the maximal ideal $\mathfrak{m}$ of A. An usual trick in reducing the problems to positive depth is the following. Given a good $\mathfrak{q}$-filtration $\mathbb{M} = \{M_n\}_{n \geq 0}$ of the d-dimensional module M, we introduce the corresponding filtration of the satureted module $M^{sat} = M/H^0_{\mathfrak{m}}(M)$ and we denote

$$\mathbb{M}^{sat} := \mathbb{M}/H^0_{\mathfrak{m}}(M) = \{M_n + H^0_{\mathfrak{m}}(M)/H^0_{\mathfrak{m}}(M)\}_{n \geq 0}.$$

In particular $\mathbb{M}^{sat}$ is a good $\mathfrak{q}$-filtration and $e_0(\mathbb{M}^{sat}) = e_0(\mathbb{M})$. If $a \in \mathfrak{q}$ is an $\mathbb{M}$-superficial element for $\mathfrak{q}$, then $a \in \mathfrak{q}$ is an $\mathbb{M}^{sat}$-superficial element for $\mathfrak{q}$.

We can relate the Hilbert coefficients of the filtrations $\mathbb{M}$ and $\mathbb{M}^{sat}$.

Proposition 2.2. *Let $\mathbb{M}$ be a good $\mathfrak{q}$-filtration of the module M and $W := H^0_{\mathfrak{m}}(M)$. Then*

$$e_i(\mathbb{M}) = e_i(\mathbb{M}^{sat}) \quad 0 \leq i \leq d-1, \qquad e_d(\mathbb{M}) = e_d(\mathbb{M}^{sat}) + (-1)^d \lambda(W).$$

A sequence of elements $a_1, \dots, a_r$ will be called a $\mathbb{M}$-*superficial sequence for* $\mathfrak{q}$ if for every $j = 1, \dots, r$ the element a_j is an $(\mathbb{M}/(a_1, \dots, a_{j-1})M)$-superficial element for $\mathfrak{q}$.

Let $a_1, \dots, a_d$ an $\mathbb{M}$-superficial sequence for $\mathfrak{q}$, then

$$M_{n+1} = JM_n \text{ for } n \gg 0. \tag{8}$$

where $J = (a_1, \dots, a_d)$. In particular $\mathbb{M}$ is also a good J-filtration. The above equality says that J is a $\mathbb{M}$-reduction of $\mathfrak{q}$.

If $\mathfrak{q}$ is $\mathfrak{m}$-primary and the residue field is infinite, there is a complete correspondence between maximal $\mathbb{M}$-superficial sequences for $\mathfrak{q}$ and minimal $\mathbb{M}$-reductions of $\mathfrak{q}$ (see for example [23]). Every minimal $\mathbb{M}$-reduction J of $\mathfrak{q}$ can be generated by a maximal $\mathbb{M}$-superficial sequence, conversely the ideal generated by a maximal $\mathbb{M}$-superficial sequence is a minimal $\mathbb{M}$-reduction of $\mathfrak{q}$.

Notice that if $\mathfrak{q} = (x^7, x^6y, x^3y^4, x^2y^5, y^7)$ in $k[[x, y]]$, then $J = (x^7, y^7)$ is a minimal reduction of $\mathfrak{q}$, but $\{x^7, y^7\}$ is not a superficial sequence. We can verify that $\{x^7 + y^7, y^7\}$ is a superficial sequence for $\mathfrak{q}$.

Given a good $\mathfrak{q}$-filtration $\mathbb{M}$ of the module M of dimension d, let $a_1, \dots, a_d$ be an $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. We denote by J the ideal they generate and consider the J-adic filtration of the module M. This is by definition the filtration

$$\mathbb{N} := \{J^j M\}_{j \geq 0}$$

which is clearly a good J-filtration. By (8) M is also a good J-filtration, so that, by (6), $e_0(\mathbb{M}) = e_0(\mathbb{N})$.

In the case M is Cohen-Macaulay, the elements $a_1, \dots, a_d$ form a regular sequence on M so that $J^iM/J^{i+1}M \simeq (M/JM)^{\binom{d+i-1}{i}}$. It implies that the Hilbert Series of $\mathbb{N}$ is $HS_{\mathbb{N}}(z) = \frac{\lambda(M/JM)}{(1-z)^d}$ and thus $e_i(\mathbb{N}) = 0$ for every $i \geq 1$. This proves that these integers give a good measure of how M differs from being Cohen-Macaulay. In the case $M = A$, Vasconcelos conjectured that if A is not Cohen-Macaulay, then $e_1(J) < 0$ for large classes of local rings.

The integer $e_1(J)$ is exactly the correction term that S. Goto and K. Nishida introduced in [15] getting free of the Cohen-Macaulyness of A.

If M is a generalized Cohen-Macaulay module of dimension ≥ 2, then

$$e_1(\mathbb{N}) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda(H^i_{\mathfrak{m}}(M))$$

with equality if M is Buchsbaum. This is due to Stuckrad and Vogel. S. Goto and K. Nishida proved the following result (see [15]).

Lemma 2.3. *Let M be a finitely generated A-module of dimension one and let a be a parameter for M. Then for every $t >> 0$ we have $H^0_{\mathfrak{m}}(M) = 0 :_M a^t$ and, if we denote by $\mathbb{N}$ the (a)-adic filtration on M, $\lambda(H^0_{\mathfrak{m}}(M)) = -e_1(\mathbb{N})$.*

Proof. Since there is an integer j such that $\mathfrak{m}^j M \subseteq aM = ((a) + 0 : M)M$, the ideal $(a) + 0 : M$ is $\mathfrak{m}$-primary and therefore $\mathfrak{m}^s \subseteq (a) + 0 : M$ for some s; this implies

$$\mathfrak{m}^{ts} \subseteq (a)^t + 0 : M$$

for every t. On the other hand, $W = 0 :_M \mathfrak{m}^t$ for every integer $t >> 0$, so that

$$W = 0 :_M \mathfrak{m}^t \subseteq 0 :_M a^t \subseteq 0 :_M \mathfrak{m}^{ts} = W.$$

We denote by $\mathbb{N}^n$ the (a^n)-adic filtration on M. Now for $n \gg 0$, it is easy to see that $ne_0(\mathbb{N}) = e_0(\mathbb{N}^n) = \lambda(M/a^nM) - \lambda(0 :_M a^n)$ and the result follows because $\lambda(M/a^nM) = ne_0(\mathbb{N}) - e_1(\mathbb{N})$. □

By using the previous fact we can prove the following results:

Theorem 2.4. *Let $\mathbb{M}$ be a good $\mathfrak{q}$-filtration of a module M of dimension d and let J be an ideal generated by a maximal sequence of $\mathbb{M}$-superficial elements for $\mathfrak{q}$; then we have*

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) \leq \sum_{j \geq 0} \lambda(M_{j+1}/JM_j).$$

In the classical case, when $M = A$ is Cohen-Macaulay and $\mathbb{M} = \{\mathfrak{q}^j\}$, the above inequality is due to S. Huckaba (see [21]). He also proved that equality holds if and only if the associated graded ring has depth at least $d-1$. We will extend this result in Theorem 2.10.

By using the same strategy we improve **Northcott's inequalty** for filtrations on a module which is not necessarily Cohen-Macaulay.

Theorem 2.5. *Let $\mathbb{M} = \{M_j\}_{j\geq 0}$ be a good $\mathfrak{q}$-filtration of a module M of dimension d and let J be an ideal generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. For every integer $s \geq 1$ we have*

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) \geq s\, e_0(\mathbb{M}) - \lambda(M/M_{s-1}) - \lambda(M/M_s + JM).$$

Remark 2.6. Let us apply the above theorem in the very particular case when $M_j = \mathfrak{q}^j$ for every $j \geq 0$ and $\mathfrak{q}$ is a primary ideal of A.

- $s = 1$

$$e_1(\mathfrak{q}) - e_1(J) \geq e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q})$$

which is exactly a result proved by S. Goto and K. Nishida. If A is Cohen-Macaulay, then we get Northcott's bound $e_1(\mathfrak{q}) \geq e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q})$. Moreover if $\mathfrak{q} = \mathfrak{m}$ then we have $e_1(\mathfrak{m}) \geq e_0(\mathfrak{m}) - 1$ already discussed in the previous section.

- $s = 2$

$$e_1(\mathfrak{q}) - e_1(J) \geq 2e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q}) - \lambda(A/\mathfrak{q}^2 + J).$$

This is a result recently proved by A. Corso (see [5]) and, if A is Cohen-Macaulay, by J. Elias and G. Valla (see [11]).

The good behaviour of the superficial elements leads us to prove Theorems 2.4 and 2.5 by induction on the dimension of M. It will be useful to remark that, since $\mathbb{M}$ and $\mathbb{N}$ are good J-filtrations, we may find $a_1, \dots, a_{d-1}$ in J superficial for both filtrations. Hence, by (7), we get

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) = e_1(\mathbb{M}/(a_1, \dots, a_{d-1})M) - e_1(\mathbb{N}/(a_1, \dots, a_{d-1})M).$$

Now $M/(a_1, \dots, a_{d-1})M$ is a 1-dimensional module. For this reason we need an *ad hoc* treatment of the one dimensional case.

Assume $\dim M = 1$, then the module $M/H^0_{\mathfrak{m}}(M)$ is Cohen-Macaulay and, by Lemma 2.3 and Proposition 2.2, we have

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) = e_1(\mathbb{M}^{sat}) \tag{9}$$

where $\mathbb{N}$ is the (a)-adic filtration on M for any $\mathbb{M}$-superficial element a for $\mathfrak{q}$. The above equality reduces the problem to a Cohen-Macaulay module of dimension one.

Let M be a **Cohen-Macaulay module of dimension one** endowed with the good $\mathfrak{q}$-filtration $\mathbb{M}$. We know that, in this case, for large n we have $e_0(\mathbb{M}) = HF_{\mathbb{M}}(n) = \lambda(M/aM)$ where a is an $\mathbb{M}$-superficial element for $\mathfrak{q}$. Let $j \geq 0$, from the diagram

$$\begin{array}{ccc} M & \supset & M_{j+1} \\ \cup & & \cup \\ aM & \supset & aM_j \end{array}$$

we deduce

$$HF_{\mathbb{M}}(j) = e_0(\mathbb{M}) - v_j(\mathbb{M}) \tag{10}$$

where $v_j(\mathbb{M}) = \lambda(M_{j+1}/aM_j)$. It is clear that $v_j(\mathbb{M}) = 0$ for $j \gg 0$. Starting from (10), we can compute the Hilbert series of M

$$HS_{\mathbb{M}}(z) = \frac{e_0 - v_0(\mathbb{M}) + \sum_{j\geq 0}(v_j(\mathbb{M}) - v_{j+1}(\mathbb{M}))z^{j+1}}{(1-z)}$$

Hence in the 1-dimensional Cohen-Macaulay case

$$e_i(\mathbb{M}) = \sum_{j \geq i-1} \binom{j}{i-1} v_j(\mathbb{M}) \tag{11}$$

for every $i \geq 1$.

In order to prove Theorems 2.4 and 2.5, it will be crucial to prove the following results in dimension one.

Proposition 2.7. *Let $\mathbb{M}$ be a good $\mathfrak{q}$-filtration of a module M of dimension one, let a be an $\mathbb{M}$-superficial element for $\mathfrak{q}$ and $\mathbb{N}$ the (a)-adic filtration on M. Then*

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) \leq \sum_{j\geq 0} \lambda(M_{j+1}/aM_j).$$

If $H^0_{\mathfrak{m}}(M) \subseteq M_1$, the equality holds if and only if M is Cohen-Macaulay.

Proof. We have $e_1(\mathbb{M}^{sat}) = \sum_{j\geq 0} \lambda(M^{sat}_{j+1}/aM^{sat}_j)$. Denote $W = H^0_{\mathfrak{m}}(M)$, now

$$\lambda(M^{sat}_{j+1}/aM^{sat}_j) = \lambda(M_{j+1}/aM_j + M_{j+1} \cap W) \leq \lambda(M_{j+1}/aM_j)$$

and the result follows by (9). The equality holds if and only if $M_{j+1} \cap W \subseteq aM_j$ for every $j \geq 0$. If M is Cohen-Macaulay then $e_1(\mathbb{N}) = 0$ and the result follows by (11). Hence we prove now that if $W \subseteq M_1$ and the equality holds, then $W = 0$. We know that $M_{j+1} \cap W \subseteq aM_j$ for every $j \geq 0$, in particular $M_1 \cap W = W \subseteq aM$. Since $W = 0 :_M a^t$ for $t \gg 0$, it is easy to see that $W \subseteq aW$. In fact if $c \in W$, then $c \in a(0 :_M a^{t+1}) = aW$. Hence by Nakayama $W = 0$, as wanted. □

Concerning the extension of Northcott's result the crucial point will be the following:

Proposition 2.8. *Let $\mathbb{M} = \{M_j\}_{j\geq 0}$ be a good $\mathfrak{q}$-filtration of a module M of dimension one and let a be an $\mathbb{M}$-superficial element for $\mathfrak{q}$. Then for every integer $s \geq 1$ and for every $n \gg 0$ we have*

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) = s\, e_0(\mathbb{M}) - \lambda(M/M_s) + \lambda(M_s + H^0_{\mathfrak{m}}(M)/M_s) + \lambda(M_n/a^{n-s}M_s).$$

Proof. We have for every $n \gg 0$ the following equalities:

$$\lambda(M/M_n) = HP^1_{\mathbb{M}}(n-1) = e_0(\mathbb{M})n - e_1(\mathbb{M})$$

$$\lambda(M/a^{n-s}M) = HP^1_{\mathbb{N}}(n-s-1) = e_0(\mathbb{N})(n-s) - e_1(\mathbb{N}).$$

Since $e_0(\mathbb{M}) = e_0(\mathbb{N})$, we get

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) = s\, e_0(\mathbb{M}) - \lambda(M/M_n) + \lambda(M/a^{n-s}M).$$

From the diagram

$$\begin{array}{ccc} M & \supset & M_n \\ \cup & & \cup \\ a^{n-s}M & \supset & a^{n-s}M_s \end{array}$$

we get

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) = s\, e_0(\mathbb{M}) + \lambda(M_n/a^{n-s}M_s) - \lambda(a^{n-s}M/a^{n-s}M_s).$$

By using the exact sequence

$$0 \to (M_s + 0 :_M a^{n-s}/M_s) \to M/M_s \stackrel{a^{n-s}}{\to} a^{n-s}M/a^{n-s}M_s \to 0$$

and the equality $0 :_M a^t = H^0_{\mathfrak{m}}(M)$ for $t >> 0$, we get the conclusion. □

We omit here the complete proof of Theorems 2.4 and 2.5 since they are now easy consequences of Propositions 2.7, 2.8 and (7).

If M is Cohen-Macaulay, then $e_1(\mathbb{N}) = 0$, hence the bound of Theorem 2.4 can be refined. Hence, by the properties of the superficial elements we can prove the following result (see [38] for more details) already proved by S. Huckaba and T. Marley for ideal filtrations.

Theorem 2.9. *Let $\mathbb{M} = \{M_j\}_{j\geq 0}$ be a good $\mathfrak{q}$-filtration of the Cohen-Macaulay module M of dimension $d \geq 1$ and let J be an ideal generated by a maximal sequence of $\mathbb{M}$-superficial elements for $\mathfrak{q}$. Then we have*

$$e_1(\mathbb{M}) \leq \sum_{j\geq 0} v_j(\mathbb{M})$$

The following conditions are equivalent

1. *depth $gr_{\mathbb{M}}(M) \geq d - 1$.*
2. *$e_1(\mathbb{M}) = \sum_{j\geq 0} v_j(\mathbb{M})$.*
3. *$HS_{\mathbb{M}}(z) = \frac{e_0(\mathbb{M})-v_0(\mathbb{M})+\sum_{j\geq 0}(v_j(\mathbb{M})-v_{j+1}(\mathbb{M}))z^{j+1}}{(1-z)^d}$.*

Consider $M = A$ and $\mathbb{M}$ the $\mathfrak{m}$-adic filtration on A. If we do not assume A is Cohen-Macaulay, surprisingly, the equality in Theorem 2.4 forces the ring A itself to be Cohen-Macaulay. By using Proposition 2.7, Theorem 2.9 and c), e) among the properties of the superficial elements, we can prove the following result.

Theorem 2.10. *Let $(A, \mathfrak{m})$ be a local ring of dimension $d \geq 1$ and let J be the ideal generated by a maximal $\mathfrak{m}$-superficial sequence. The following conditions are equivalent:*

1. *$e_1(\mathfrak{m}) - e_1(J) = \sum_{j\geq 0} v_j(\mathfrak{m})$.*
2. *A is Cohen-Macaulay and depth $gr_{\mathfrak{m}}(A) \geq d - 1$.*

3 Applications to the Sally module and to the Fiber Cone

We show that the Fiber cone $F_{\mathfrak{m}}(\mathfrak{q})$ and the Sally-module $S_J(\mathfrak{q})$ of an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ fit into suitable short exact sequences, together with algebras associated to filtrations. The aim of this section is to take advantage of this for giving shorter proofs of selected results of the recent literature. We can study the Hilbert function and the depth of these algebras by using the knowledge and the methods already developed in Section 2.

The Fiber Cone

Let $(A, \mathfrak{m})$ be a commutative local ring of dimension d and let $\mathfrak{q}$ be an ideal of A. We define the graded $gr_{\mathfrak{q}}(A) = \oplus_{n\geq 0}\mathfrak{q}^n/\mathfrak{q}^{n+1}$-module

$$F_{\mathfrak{m}}(\mathfrak{q}) = \oplus_{n\geq 0}\mathfrak{q}^n/\mathfrak{m}\mathfrak{q}^n$$

which is called the Fiber cone of $\mathfrak{q}$. It coincides with $gr_{\mathfrak{m}}(A)$ in the case $\mathfrak{q} = \mathfrak{m}$.

This graded object encodes several information on $\mathfrak{q}$. For instance, its dimension coincides with the minimal number of generators of any minimal reduction of $\mathfrak{q}$, that is the analytic spread of $\mathfrak{q}$ and its Hilbert function controls the minimal number of generators of the powers of $\mathfrak{q}$.

Usually the arithmetical properties of the Fiber cone and those of the associated graded ring have been studied apparently with different approaches. The literature concerning the associated graded rings is much more rich, but new and peculiar tecniques had been necessary in order to study several problems on the Fiber cone.

Papers by T. Cortadellas and S. Zarzuela (see [7]) proved the existence of an exact sequence of the homology of modified Koszul complexes, relating $F_{\mathfrak{m}}(\mathfrak{q})$ with the associated graded modules to the $\mathfrak{q}$-adic filtration and the $\mathfrak{q}$-good filtration

$$\mathbb{F} := \{\mathfrak{m}\mathfrak{q}^n\} : \quad A \supseteq \mathfrak{m} \supseteq \mathfrak{m}\mathfrak{q} \supseteq \mathfrak{m}\mathfrak{q}^2 \supseteq \dots$$

But this idea has not been exploited so deeper. Starting from their work, G. Valla and myself have developed this strategy.

First we prove some results concerning the depths of $F_{\mathfrak{m}}(\mathfrak{q})$, $gr_{\mathfrak{q}}(A)$ and $gr_{\mathbb{F}}(A)$. Since the involved objects are graded modules on $gr_{\mathfrak{q}}(A)$, the depths are computed with respect to $Q = \oplus_{n>0}\mathfrak{q}^n/\mathfrak{q}^{n+1}$.

Proposition 3.1. *We have the following homogeneous exact sequences of $gr_{\mathfrak{q}}(A)$-graded modules:*

$$0 \to N \to \ gr_{\mathfrak{q}}(A) \longrightarrow \ F_{\mathfrak{m}}(\mathfrak{q}) \to 0$$

$$0 \to \ F_{\mathfrak{m}}(\mathfrak{q}) \to \ gr_{\mathbb{F}}(A) \longrightarrow N(-1) \to 0$$

where $N = \oplus_{n \geq 0}\mathfrak{m}M_n/M_{n+1}$. Let p be an integer, we have

1. depth $F_{\mathfrak{m}}(\mathfrak{q}) \geq min\{$depth $gr_{\mathfrak{q}}(A) + 1,$ depth $gr_{\mathbb{F}}(A)\}$
2. *If $min\{$*depth *$gr_{\mathfrak{q}}(A),$* depth *$F_{\mathfrak{m}}(\mathfrak{q})\} \geq p$, then* depth $gr_{\mathbb{F}}(A) \geq p$.
3. *If $min\{$*depth *$gr_{\mathbb{F}}(A),$* depth *$F_{\mathfrak{m}}(\mathfrak{q})\} \geq p$, then* depth$gr_{\mathfrak{q}}(A) \geq p - 1$.

Proof. It is enough to remark that we have the exact sequences of the corresponding homogeneous parts of degree n of the involved graded algebras. Then the information on depths comes from "depth's formula". □

Several examples show that $F_{\mathfrak{m}}(\mathfrak{q})$ can be Cohen-Macaualy even if $\mathrm{gr}_{\mathfrak{q}}(A)$ is not Cohen-Macaulay and conversely. The above proposition clarifies the intermediate role of the graded module associated to the filtration $\mathbb{F}$.

It will be important to remind that, as we have seen, it is possible to find a superficial sequence $a_1, \dots, a_r$ in $\mathfrak{q}$ which is both superficial for the $\mathfrak{q}$-adic filtration and $\mathbb{F}$-superficial for $\mathfrak{q}$.

As a consequence of the above proposition, immediately we reprove several results which are known in the literacture. The following result was already proved by K. Shah and by several other authors by different methods. We give here a short proof.

Theorem 3.2. *Let $\mathfrak{q}$ be an ideal of a local ring $(A, \mathfrak{m})$ and let J be an ideal generated by a superficial regular sequence for $\mathfrak{q}$ such that $\mathfrak{q}^2 = J\mathfrak{q}$. Then $F_{\mathfrak{m}}(\mathfrak{q})$ is Cohen-Macaulay.*

Proof. By using the assumption, we get that $\mathfrak{q}^{n+1} \cap J = J\mathfrak{q}^n$ and $\mathfrak{m}\mathfrak{q}^{n+1} \cap J = J\mathfrak{m}\mathfrak{q}^n$ for every integer n. By Valabrega-Valla criterion it follows that the filtrations $\{\mathfrak{q}^n\}$ and $\{\mathfrak{m}\mathfrak{q}^n\}$ on A have associated graded ring of depth at least $\mu(J)$ =dim $F_{\mathfrak{m}}(\mathfrak{q})$. The result follows now by Proposition 3.1. (1.). □

For every ideal $\mathfrak{q} \neq 0$, we may define the numerical function

$$HF_{F_{\mathfrak{m}}(\mathfrak{q})}(n) := \dim_k(\mathfrak{q}^n/\mathfrak{m}\mathfrak{q}^n) = \mu(\mathfrak{q}^n)$$

which is the Hilbert function of $F_{\mathfrak{m}}(\mathfrak{q})$. As usual we denote by $HS_{F_{\mathfrak{m}}(\mathfrak{q})}(z)$ its generating Hilbert series.

We remark that, under the assumptions of Theorem 3.2, we can easily write the Hilbert series of $F_{\mathfrak{m}}(\mathfrak{q})$ which is a standard graded k-algebra. In fact

$$HS_{F_{\mathfrak{m}}(\mathfrak{q})}(z) = \frac{1}{(1-z)^d} HS_{F_{\mathfrak{m}}(\mathfrak{q})/JF_{\mathfrak{m}}(\mathfrak{q})}(z) = \frac{1}{(1-z)^d} \sum_{i \geq 0}^{s} \lambda(\mathfrak{q}^i / J\mathfrak{q}^{i-1} + \mathfrak{m}\mathfrak{q}^i) z^i$$

Since $\mathfrak{q}^2 = J\mathfrak{q}$ and $\lambda(\mathfrak{q}/J + \mathfrak{m}\mathfrak{q}) = \mu(\mathfrak{q}) - d$, one has

$$HS_{F_{\mathfrak{m}}(\mathfrak{q})}(z) = \frac{1 + (\mu(\mathfrak{q}) - d)z}{(1-z)^d}.$$

From now on assume that $\mathfrak{q}$ is $\mathfrak{m}$-primary, then $\dim F_{\mathfrak{m}}(\mathfrak{q}) = d = \dim A$. We recall that $HF_{F_{\mathfrak{m}}(\mathfrak{q})}(n)$ is a polynomial function and the corresponding polynomial $HP_{F_{\mathfrak{m}}(\mathfrak{q})}(X)$ has degree $d-1$. It is the Hilbert polynomial of $F_{\mathfrak{m}}(\mathfrak{q})$ and, as usual, we can write

$$HP_{F_{\mathfrak{m}}(\mathfrak{q})}(X) = \sum_{i=0}^{d-1} (-1)^i f_i(\mathfrak{q}) \binom{X + d - i - 1}{d - i - 1}.$$

The coefficients $f_i(\mathfrak{q})$ are integers and they are called the Hilbert coefficients of $F_{\mathfrak{m}}(\mathfrak{q})$. In particular $f_0(\mathfrak{q})$ is the multiplicity of the fiber cone. We remark that A.V. Jayanthan and J. Verma introduced different coefficients for $F_{\mathfrak{m}}(\mathfrak{q})$ (called $g_i(\mathfrak{q})$), obtained by splitting the Hilbert polynomial of $gr_{\mathbb{F}}(A)$ with respect to a special basis. Similar results can be obtained in this case (see [38] for results concerning the $g_i's$).

From the exact sequences, we can relate the Hilbert function of $F_{\mathfrak{m}}(\mathfrak{q})$ with those of the associated graded rings $gr_{\mathfrak{q}}(A)$ and $gr_{\mathbb{F}}(A)$. Since $e_0(\mathfrak{q}) = e_0(\mathbb{F})$, a computation shows that

$$f_{i-1}(\mathfrak{q}) = e_i(\mathfrak{q}) + e_{i-1}(\mathfrak{q}) - e_i(\mathbb{F}) \tag{12}$$

for every $i = 1, \ldots, d$.

It is clear now that the theory developed on the Hilbert coefficients of the graded module associated to a good filtration on A can be applied to $e_i(\mathfrak{q})$ and $e_i(\mathbb{F})$ in order to get information, through (12), on the coefficients of the Fiber cone of $\mathfrak{q}$. We present here a couple of results which have been obtained in the literature by different methods. The following is an extension to modules of a recent result by A. Corso (see [5]).

Theorem 3.3. *Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of a local ring $(A, \mathfrak{m})$ of dimension d. Let J be the ideal generated by a maximal superficial sequence for $\mathfrak{q}$, then*

$$f_0(\mathfrak{q}) \leq min\{e_1(\mathfrak{q}) - e_0(\mathfrak{q}) - e_1(J) + \lambda(A/\mathfrak{q}) + \mu(\mathfrak{q}) - d + 1, e_1(\mathfrak{q}) - e_1(J) + 1\}.$$

Proof. Since $f_0(\mathfrak{q}) = e_0(\mathfrak{q}) + e_1(\mathfrak{q}) - e_1(\mathbb{F})$ by (12), it is enough to apply Theorem 2.5 to $e_1(\mathbb{F})$ for $s = 1, 2$.

□

If A is Cohen-Macaulay, then $e_1(J) = 0$ because J is generated by a regular sequence and we are able to control the extremal cases. By using our approach we can reprove the following result by A. Corso, C. Polini and W. Vasconcelos (see [4]).

Corollary 3.4. *Let $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal of a local Cohen-Macaulay ring $(A, \mathfrak{m})$ of dimension d. Then*

$$f_0(\mathfrak{q}) \le e_1(\mathfrak{q}) - e_0(\mathfrak{q}) + \lambda(A/\mathfrak{q}) + \mu(\mathfrak{q}) - d + 1 \le e_1(\mathfrak{q}) + 1.$$

In particular

1. *If $f_0(\mathfrak{q}) = e_1(\mathfrak{q}) + 1$, then $\mathfrak{m}\mathfrak{q} = \mathfrak{m}J$ for every maximal superficial sequence J for $\mathfrak{q}$. If, in addition, $\lambda(\mathfrak{q}^2 \cap J/J\mathfrak{q}) \le 1$ for some J, then depth $gr_{\mathfrak{q}}(A) \ge r - 1$ and $F_{\mathfrak{m}}(\mathfrak{q})$ is Cohen-Macaulay.*

2. *If $f_0(\mathfrak{q}) = e_1(\mathfrak{q}) - e_0(\mathfrak{q}) + \lambda(A/\mathfrak{q}) + \mu(\mathfrak{q}) - r + 1$, then $F_{\mathfrak{m}}(\mathfrak{q})$ is unmixed.*

Proof. The first inequality follows by Theorem 3.3. We prove now that $e_1(\mathfrak{q}) - e_0(\mathfrak{q}) + \lambda(A/\mathfrak{q}) + \nu(\mathfrak{q}) - r + 1 \le e_1(\mathfrak{q}) + 1$. Indeed, if J is an ideal generated by a maximal superficial sequence for $\mathfrak{q}$, then $e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q}) - \nu(\mathfrak{q}) + r = \lambda(\mathfrak{q}/J) - \lambda(\mathfrak{q}/\mathfrak{q}\mathfrak{m}) + \lambda(J/J\mathfrak{m}) = \lambda(\mathfrak{q}/J\mathfrak{m}) - \lambda(\mathfrak{q}/\mathfrak{q}\mathfrak{m}) \ge 0$.

If $f_0(\mathfrak{q}) = e_1(\mathfrak{q}) + 1$, it turns out $\mathfrak{q}\mathfrak{m} = J\mathfrak{m}$ and hence the associated graded module to the filtration $\mathbb{F} = \{\mathfrak{m}\mathfrak{q}^n\}$ is Cohen-Macaulay by Valabrega-Valla criterion. Now $\mathfrak{q}^2 \subseteq \mathfrak{m}\mathfrak{q} = \mathfrak{m}J \subseteq J$, then $\lambda(\mathfrak{q}^2 \cap J/J\mathfrak{q}) = \lambda(\mathfrak{q}^2/J\mathfrak{q}) \le 1$. Hence by Corollary 1.7 in [35], it is known that depth $gr_{\mathfrak{q}}(A) \ge d - 1$ and 1. follows by Proposition 3.1.

Now, from the proof of Theorem 3.3, $f_0(\mathfrak{q}) = e_1(\mathfrak{q}) - e_0(\mathfrak{q}) + \lambda(A/\mathfrak{q}) + \nu(\mathfrak{q}) - r + 1$ if and only if $e_1(\mathbb{F}) = 2e_0(\mathfrak{q}) - 1 - \lambda(A/\mathfrak{m}\mathfrak{q} + J)$ (equality in Northcott's bound with $s = 2$) and hence, by a result by J.Elias and G. Valla in [11] stated for filtration, $gr_{\mathbb{F}}(A)$ is Cohen-Macaulay. Because we have a canonical injective map from $F_{\mathfrak{m}}(\mathfrak{q})$ to $gr_{\mathbb{F}}(A)$ the result follows. □

Notice that the assumption $\lambda(\mathfrak{q}^2 \cap J/J\mathfrak{q}) \le 1$ is satisfied for instance if A is Gorenstein.

The Sally module

Given an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ in the local ring $(A, \mathfrak{m})$ and a minimal reduction J of $\mathfrak{q}$, W. Vasconcelos introduced the so-called **Sally module** $S_J(\mathfrak{q})$ of $\mathfrak{q}$ with respect to J. It is a $\mathcal{R}(J) = \oplus_i J^i$-module defined by the exact sequence

$$0 \to \mathfrak{q}A[Jt] \to \mathfrak{q}A[\mathfrak{q}t] \to S_J(\mathfrak{q}) = \oplus_{n\ge1}\mathfrak{q}^{n+1}/J^n\mathfrak{q} \to 0.$$

Assume $\mathfrak{q}$ is $\mathfrak{m}$-primary, the Hilbert function of this graded module is

$$HF_{S_J(\mathfrak{q})}(n) := \lambda(q^{n+1}/J^n\mathfrak{q}),$$

and its Hilbert series is $HS_{S_J(\mathfrak{q})}(z) = \sum_{n\ge1}\lambda(q^{n+1}/J^n\mathfrak{q})z^n$. We write $e_i(S_J(\mathfrak{q}))$ the corresponding Hilbert coefficients. W. Vasconcelos and M. Vaz Pinto proved that if A is Cohen-Macaulay, then

$$e_0(S_J(\mathfrak{q})) \le \sum_{j\ge1}\lambda(\mathfrak{q}^{j+1}/J\mathfrak{q}^j)(= v_j(\mathfrak{q}))$$

and equality holds if and only if depth $gr_{\mathfrak{q}}(A) = \oplus_{n\ge0}\mathfrak{q}^n/\mathfrak{q}^{n+1} \ge d - 1$.

As an application of the results of Section 2., in this section we extend the above inequality without assuming the Cohen-Macaulayness of A and we study the extremal case.

Let $\mathbb{M}$ be a good $\mathfrak{q}$-filtration of the A-module M of dimension d and let J be the ideal generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. We consider the filtration

$$\mathbb{E} := A \supseteq \mathfrak{q} \supseteq J\mathfrak{q} \supseteq J^2\mathfrak{q} \supseteq \cdots \supseteq J^n\mathfrak{q} \supseteq \dots$$

which is a good J-filtration with the nice property that $E_{n+1} = JE_n$ for all $n \ge 1$. Then $S_J(\mathfrak{q})$ is related to $gr_{\mathfrak{q}}(A)$ and $gr_{\mathbb{E}}(A)$ (graded $A[Jt]$-modules) by the following two short exact sequences:

$$0 \to J^{n-1}\mathfrak{q}/J^n\mathfrak{q} \to \mathfrak{q}^n/J^n\mathfrak{q} \to \mathfrak{q}^n/J^{n-1}\mathfrak{q} \to 0$$

$$0 \to \mathfrak{q}^{n+1}/J^n\mathfrak{q} \to \mathfrak{q}^n/J^n\mathfrak{q} \to \mathfrak{q}^n/\mathfrak{q}^{n+1} \to 0.$$

Since $\mathfrak{q}^n/J^{n-1}\mathfrak{q} = (S_J(\mathfrak{q})(-1))_n$, by standard facts it follows that

$$\text{depth } gr_{\mathfrak{q}}(A) \geq \min\{\text{depth } S_J(\mathfrak{q}) - 1, \text{depth } gr_{\mathbb{E}}(A)\} \tag{13}$$

Moreover we get $HS_{S_J(\mathfrak{q})(-1)}(z) + HS_{\mathbb{E}}(z) = HS_{\mathfrak{q}}(z) + HS_{S_J(\mathfrak{q})}(z)$ so that

$$(z-1)HS_{S_J(\mathfrak{q})}(z) = HS_{\mathfrak{q}}(z) - HS_{\mathbb{E}}(z) \tag{14}$$

We deduce that $\dim S_J(\mathfrak{q}) = d$ if and only if $e_1(\mathfrak{q}) > e_1(\mathbb{E})$. If $\dim S_J(\mathfrak{q}) = d$, then for every $i \geq 0$ we have

$$\boxed{e_i(S_J(\mathfrak{q})) = e_{i+1}(\mathfrak{q}) - e_{i+1}(\mathbb{E})} \tag{15}$$

Now, if we apply Theorem 2.5 with $\mathbb{M} = \mathbb{E}$ and $s = 1$, we get

$$e_1(\mathbb{E}) - e_1(J) \geq e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q}),$$

then

$$e_0(S_J(\mathfrak{q})) = e_1(\mathfrak{q}) - e_1(\mathbb{E}) \leq e_1(\mathfrak{q}) - e_1(J) - e_0(\mathfrak{q}) + \lambda(A/\mathfrak{q}). \tag{16}$$

Notice that the above inequality has been recently proved by A. Corso (see [5]) and it extends a result proved by W. Vasconcelos in the Cohen-Macaulay case ($e_1(J) = 0$).

In (16) we can bound $e_1(\mathfrak{q})$ by Theorem 2.4 and we get

$$e_0(S_J(\mathfrak{q})) \leq \sum_{j\geq 0} v_j(\mathfrak{q}) - e_0(\mathfrak{q}) + \lambda(A/\mathfrak{q}). \tag{17}$$

The equality holds if and only if

We prove the following result which completes Theorem 2.10.

Theorem 3.5. *Let $(A, \mathfrak{m})$ be a local ring of dimension $d \geq 1$ and let J be an ideal generated by a maximal $\mathfrak{m}$-superficial sequence. If $\dim S_J(\mathfrak{m}) = d$, then $e_0(S_J(\mathfrak{m})) \leq \sum_{j\geq 0} v_j(\mathfrak{m}) - e_0(\mathfrak{m}) + 1$. Moreover the following conditions are equivalent:*

1. $e_0(S_J(\mathfrak{m})) = \sum_{j\geq 0} v_j(\mathfrak{m}) - e_0(\mathfrak{m}) + 1$
2. $e_1(\mathfrak{m}) - e_1(J) = \sum_{j\geq 0} v_j(\mathfrak{m})$
3. *A is Cohen-Macaulay and depth $gr_{\mathfrak{m}}(A) \geq d - 1$.*

Proof. In (17) the equality holds if and only if $e_1(\mathfrak{q}) - e_1(J) = \sum_{j\geq 0} v_j(\mathfrak{q})$. Hence, in the case of the $\mathfrak{m}$-adic filtration on A, by Theorem 2.10, the equality is equivalent to have A Cohen-Macaulay and depth $gr_{\mathfrak{m}}(A) \geq d - 1$. □

The following result completes Theorem 3.5 in the case A is Cohen-Macaulay and it reproves a series of results proved by M. Vaz Pinto in [52].

First we remark that if A is Cohen-Macaulay, then $gr_{\mathbb{E}}(A)$ is Cohen-Macaulay with minimal multiplicity ($E_2 = JE_1$) and

$$HS_{\mathbb{E}}(z) = \frac{\lambda(A/\mathfrak{q}) + (e_0(\mathbb{M}) - \lambda(A/\mathfrak{q}))z}{(1-z)^d}.$$

In particular $e_1(\mathbb{E}) = e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q})$.

By using (15), (13), (14) we get

1. If $\dim S_J(\mathfrak{q}) = d$, then $e_0(S_J(\mathfrak{q})) = e_1(\mathfrak{q}) - e_0(\mathfrak{q}) + \lambda(A/\mathfrak{q})$
2. depth $gr_{\mathfrak{q}}(A) \geq$ depth $S_J(\mathfrak{q}) - 1$
3. $(z-1)HS_{S_J(\mathfrak{q})}(z) = HS_{\mathfrak{q}}(z) - \frac{\lambda(A/\mathfrak{q})+(e_0(\mathfrak{q})-\lambda(A/\mathfrak{q}))z}{(1-z)^d}$.

We recall that 3. says in particular that the Hilbert function of the Sally module is not decreasing.

The assumption $\dim S_J(\mathfrak{q}) = d$ is equivalent to $S_J(\mathfrak{q}) \neq 0$. In fact, we have that $\dim S_J(\mathfrak{q}) = d$ if and only if $e_1(\mathfrak{q}) > e_1(\mathbb{E}) = e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q})$. This is equivalent to $\mathfrak{q}^2 \neq J\mathfrak{q}$ and hence $S_J(\mathfrak{q}) \neq 0$.

Remark 3.6. If $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal of a a local Cohen-Macaulay ring $(A, \mathfrak{m})$ of dimension d, the value of $e_1(\mathfrak{q})$ strongly influences the structure of the Sally module. If $e_1(\mathfrak{q}) = e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q})$, then $S_J(\mathfrak{q})$ is the trivial module. The case $e_1(\mathfrak{q}) = e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q}) + 1$ is much more difficult. S. Goto, K. Nishida, K. Ozeki proved that in this case $S_J(\mathfrak{q})$ has a very nice structure (see [16], Theorem 1.2) and, by using this surprising information and (14) we obtain

$$HS_{\mathfrak{q}}(z) = \frac{\lambda(A/\mathfrak{q}) + (e_0(\mathfrak{q}) - \lambda(A/\mathfrak{q}) - c)z + \sum_{i=2}^{c+1}(-1)^i\binom{c+1}{i}z^i}{(1-z)^d}$$

where $c = \lambda(\mathfrak{q}^2/J\mathfrak{q})$. Very recently S. Goto and K. Ozeki announced an extension of the above result where the assumption A Cohen-Macaulay is relaxed (see [17]).

Theorem 3.7. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal and let J be an ideal generated by a maximal superficial sequence for $\mathfrak{q}$. The following conditions are equivalent :*

1. $e_0(S_J(\mathfrak{q})) = \sum_{j\geq 1} v_j(\mathfrak{q})$
2. $HS_{S_J(q)}(z) = \frac{\sum_{j\geq 1} v_j(\mathfrak{q})z^j}{(1-z)^d}$
3. $S_J(\mathfrak{q})$ *is Cohen-Macaulay*

and each of them is equivalent to the equivalent conditions of Theorem 2.9.

Proof. We have $e_0(S_J(\mathfrak{q})) = e_1(\mathfrak{q}) - e_0(\mathfrak{q}) + \lambda(A/\mathfrak{q})$. Hence, by Theorem 2.9, we get

$$e_0(S_J(\mathfrak{q})) \leq \sum_{j\geq 0} v_j(\mathfrak{q}) - e_0(\mathfrak{q}) + \lambda(A/\mathfrak{q}) = \sum_{j\geq 1} v_j(\mathfrak{q}).$$

By Theorem 2.9, the equality holds if and only if $e_1(\mathfrak{q}) = \sum_{j\geq 0} v_j(\mathfrak{q})$. Hence 1. is equivalent to *1., 2., 3.* of Theorem 2.9. Because *1.* is equivalent to

$$HS_{\mathfrak{q}}(z) = \frac{\lambda(A/\mathfrak{q}) + \sum_{j\geq 0}(v_j(\mathfrak{q}) - v_{j+1}(\mathfrak{q}))z^{j+1}}{(1-z)^d}$$

and we know that

$$(z-1)HS_{S_J(\mathfrak{q})}(z) = HS_{\mathfrak{q}}(z) - \frac{\lambda(A/\mathfrak{q}) + (e_0(\mathbb{M}) - (\lambda(A/\mathfrak{q}))z}{(1-z)^d},$$

it is easy to see that *1.* is also equivalent to *2.* Now, since depth $gr_{\mathfrak{q}}(A) \geq$ depth $S_J(\mathfrak{q}) - 1$, we get that *3.* implies depth $gr_{\mathfrak{q}}(A) \geq d-1$ which is equivalent to *1.*
We have only to prove that *2.* implies *3.* We may assume $S_J(\mathfrak{q})$ of dimension d and recall that $S_J(\mathfrak{q})$ is a $\mathbb{R}(J) = A[JT]$-module and we have $S_J(\mathfrak{q})/JTS_J(\mathfrak{q}) = \oplus_{n\geq 1}\mathfrak{q}^{n+1}/J\mathfrak{q}^n$. By *2.* we deduce that $HS_{S_J(\mathfrak{q})}(z) = \frac{1}{(1-z)^d}HS_{S_J(\mathfrak{q})/JTS_J(\mathfrak{q})}(z)$. Then JT is generated by a regular sequence of lenght d =dim$S_J(\mathfrak{q})$ and hence $S_J(\mathfrak{q})$ is Cohen-Macaulay. □

4 Minimal free resolution of a module over a regular local ring

Let $(R, \mathfrak{n})$ be a regular local ring with infinite residue field k. Let I be an ideal of R and consider the local ring $A = R/I$ with maximal ideal $\mathfrak{m} = \mathfrak{n}/I$.

The aim of this section is to give information on the Betti numbers of A as R-module. If J is an homogeneous ideal in a polynomial ring P, the Hilbert function of P/J can be computed from the graded Betti numbers of a minimal P-free resolution of P/J. In the local setting we are dealing with total Betti numbers of A and the numerical invariants of a R-free resolution does not seem related to the Hilbert function. We will see that relevant information can be deduced from the free resolution of the corresponding associated graded ring $gr_{\mathfrak{m}}(A) = \oplus_{t\geq 0}\mathfrak{m}^t/\mathfrak{m}^{t+1}$. For more details on this section we refer to [39].

If dim$R = n$, then the associated graded ring $gr_{\mathfrak{n}}(R)$ with respect to the $\mathfrak{n}$-adic filtration is the polynomial ring $P = k[x_1, \dots, x_n]$. If x is a non-zero element of R, we denote by x^* (or $gr_{\mathfrak{n}}(x)$) the initial form of x in P. If $x = 0$, then $x^* = 0$. We recall that

$$gr_{\mathfrak{m}}(A) = \oplus_{i\geq 0}\mathfrak{m}^i/\mathfrak{m}^{i+1} = gr_{\mathfrak{n}}(R)/I^* = P/I^*.$$

where I^* is the homogeneous ideal generated by the initial forms of the elements of I.

Problem: Compare the numerical invariants of the R-free minimal resolution of A with those of the P-free minimal graded resolution of $gr_{\mathfrak{m}}(A)$:

$$0 \to R^{\beta_h(I)} \to R^{\beta_{h-1}(I)} \to \cdots \to R^{\beta_0(I)} \to I \to 0$$

$$0 \to P^{\beta_s(I^*)} \to P^{\beta_{s-1}(I^*)} \to \cdots \to P^{\beta_0(I^*)} \to I^* \to 0$$

The following example shows the difficulty of the problem.

Example 4.1. (see [19]) Consider $I = (x^3 - y^7, x^2y - xt^3 - z^6)$ in $R = k[[x, y, z, t]]$. Since I is a complete intersection, then a minimal free resolution of I is given by: $0 \to R \to R^2 \to I \to 0$. But we can verify that $I^* = (x^3, x^2y, x^2t^3, xt^6, x^2z^6, xy^9 - xz^6t^3, xy^8t^3, y^7t^9)$, hence $\mu(I^*) = 8$ and a minimal free resolution of I^* is given by

$$0 \to P \to P^6 \to P^{12} \to P^8 \to I^* \to 0$$

In particular depth $A = 2$ and depth $gr_{\mathfrak{m}}(A) = 0$.

Example 4.2. The Betti numbers of A and those of $gr_{\mathfrak{m}}(A)$ do not necessarily coincide even if $\mu(I) = \mu(I^*)$ and depth $A =$ depth $gr_{\mathfrak{m}}(A)$. This is the case if we consider $A = k[[t^{19}, t^{26}, t^{34}, t^{40}]]$.

The problem can be presented in the case of a filtered R-module M. Let $\mathbb{M}$ be a good $\mathfrak{n}$-filtration and consider the associated graded module $gr_{\mathbb{M}}(M) := \oplus_{n\geq 0}M_n/M_{n+1}$. It is a $gr_{\mathfrak{n}}(R) = P$-module and we are interested to compare a minimal R-free resolution of M with a minimal P-free resolution of $gr_{\mathbb{M}}(M)$, in particular we want to compare the Betti numbers $\beta_i(M)$, the homological dimension $hd_R(M)$ and the depth of M with those of $gr_{\mathbb{M}}(M)$).

We shall prove

- $\beta_i(M) \leq \beta_i(gr_{\mathbb{M}}(M))$
- $hd_R(M) \leq hd_P(gr_{\mathbb{M}}(M))$
- depth $M \geq$depth $(gr_{\mathbb{M}}(M))$

Definition. A filtered module M is said to be of *homogeneous type with respect to the filtration* $\mathbb{M}$ if

$$\beta_i(gr_{\mathbb{M}}(M)) = \beta_i(M) \text{ for every } i \geq 0$$

We say that M is of homogeneous type if M is of homogeneous type with respect to the $\mathfrak{n}$-adic filtration.

We give the following examples of modules of homogeneous type

- Let I be an ideal of R generated by a super-regular sequence. Then both $A = R/I$ and I are of homogeneous type (see [19]).
- Let I be an ideal of the regular ring $(R, \mathfrak{n})$ such that $A = R/I$ is Cohen-Macaulay of minimal multiplicity. Then I is of homogeneous type.
- Let I be an ideal of the regular ring $(R, \mathfrak{n})$ 2-generated, then I is of homogeneous type (see [19])
- Let I be an ideal of the regular ring $(R, \mathfrak{n})$ such that $\mu(I) = \mu(I^*)$ and I^* is a componentwise linear ideal. Then $A = R/I$ is of homogeneous type (see [39]).
- Let I be a Koszul module, then I is of homogeneous type (see [HI]).

If $m \in M \setminus \{0\}$, we denote by $\nu_{\mathbb{M}}(m)$ the largest integer p such that $m \in M_p$ (the so-called valuation of m with respect to $\mathbb{M}$) and we denote by m^* or $gr_{\mathbb{M}}(m)$ the residue class of m in M_p/M_{p+1} where $p = \nu_{\mathbb{M}}(m)$ and call it the initial form of m with respect to $\mathbb{M}$. If $m = 0$, we set $\nu_{\mathbb{M}}(m) = +\infty$. We will write $\nu_{\mathfrak{n}}(m)$ if we denote the valuation with respect to the $\mathfrak{n}$-adic filtration.

We recall that if N is a submodule of M, then $gr_{\mathbb{M}}(N)$ is generated by the elements x^* with $x \in N$, we write

$$gr_{\mathbb{M}}(N) = < x^* \ : \ x \in N > .$$

Given a filtered module M, we recall that an element $g \in M$ is a *lifting* of an element $h \in gr_{\mathbb{M}}(M)$ if $g^* = h$. The morphism of filtered modules $f : M \to N$ ($f(M_p) \subseteq N_p$ for every p) clearly induces a morphism of graded $gr_{\mathfrak{n}}(R)$-modules

$$gr(f) : gr_{\mathbb{M}}(M) \to gr_{\mathbb{N}}(N).$$

It is clear that $gr(\dot{})$ is a functor from the category of filtered R-modules into the category of the graded $gr_{\mathfrak{n}}(R)$-modules. Furthermore, we have a canonical embedding $gr_{\mathbb{M}}(Kerf) \to Ker(gr(f))$, but $gr(\dot{})$ in general is not exact. We will give a characterization of the exactness in the case of a complex of free R-modules. Let $F = \oplus_{i=1}^s Re_i$ be a free R-module of rank s and $\nu_1, \ldots, \nu_s$ be integers. We define the filtration $\mathbb{F} = \{F_p : p \in \mathbf{Z}\}$ on F as follows

$$F_p := \oplus_{i=1}^s \mathfrak{n}^{p-\nu_i} e_i = \{(a_1, \ldots, a_s) : a_i \in \mathfrak{n}^{p-\nu_i}\}.$$

From now on we denote the filtered free R-module F by $\oplus_{i=1}^s R(-\nu_i)$ and we call it *special filtration* on F. If $(\mathbf{F}., \delta.)$ is a complex of finitely generated free R-modules, a special filtration on $\mathbf{F.}$ is a special filtration on each F_i that makes $(\mathbf{F}., \delta.)$ a filtered complex (complex of filtered modules).

Let M be a filtered R-module finitely generated and let $S = \{f_1, \ldots, f_s\}$ be a system of elements of M and $\nu_{\mathbb{M}}(f_i)$ be the corresponding valuations. As before let $F = \oplus_{i=1}^s Re_i$ be a free R-module of rank s equipped with the filtration $\mathbb{F}$ where $\nu_i = \nu_{\mathbb{M}}(f_i)$. Then we denote the filtered free R-module F by $\oplus_{i=1}^s R(-\nu_{\mathbb{M}}(f_i))$, hence $\nu_{\mathbb{F}}(e_i) = \nu_{\mathbb{M}}(f_i)$. Let $\phi : F \to M$ be a morphism of filtered R-modules defined by

$$\phi(e_i) = f_i.$$

If we denote by $Syz(S)$ the submodule of L generated by the first syzygies of $f_1, \dots, f_s$, then $\mathrm{Syz}(S) = \mathrm{Ker}\phi$. The following diagram can help to visualize the characterization of the standard bases which will follow.

$$\begin{array}{ccccccc}
0 & \longrightarrow & Syz(S) = Ker(\phi) & \longrightarrow & F = \oplus_{i=1}^{s} R(-\nu_{\mathbb{M}}(f_i)) & \xrightarrow{\phi} & M \\
 & & & & & & e_i \longrightarrow f_i \\
 & & \uparrow & & \downarrow gr_{\mathbb{F}} & & \downarrow gr_{\mathbb{M}} \\
0 & \longrightarrow & \mathrm{Syz}(\langle f_1^*, ..., f_s^* \rangle) = Ker(gr(\phi)) & \longrightarrow & \oplus_{i=1}^{s} P(-\nu_{\mathbb{M}}(f_i)) & \xrightarrow{gr(\phi)} & gr_{\mathbb{M}}(M) \\
 & & & & & & e_i \longrightarrow f_i^*
\end{array}$$

Definition 4.3. *Let M be a filtered module. A subset $S = \{f_1, .., f_s\}$ of M is called a standard basis of M if $gr_{\mathbb{M}}(M) = \langle f_1^*, ..., f_s^* \rangle$.*

By following the initial idea of L. Robbiano and G. Valla, recently T. Shibuta (see [45]) characterized the standard bases of a filtered module as follows:

Theorem 4.4. *Let M be a filtered R-module, $f_1, \dots, f_s \in M$ and $S = \{f_1, \dots, f_s\}$. The following facts are equivalent:*

1. *$\{f_1, \dots, f_s\}$ is a standard basis of M.*
2. *$\{f_1, \dots, f_s\}$ generates M and every element of $\mathrm{Syz}(\langle f_1^*, ..., f_s^* \rangle)$ can be lifted to an element in $Syz(S)$.*
3. *$\{f_1, \dots, f_s\}$ generates M and $\mathrm{Syz}(\langle f_1^*, ..., f_s^* \rangle) = gr_{\mathbb{F}}(\mathrm{Syz}(S))$.*

Theorem 4.5. *Let M be a filtered R-module and $(\mathbf{G}., d.)$ a P-free graded resolution of $gr_{\mathbb{M}}(M)$. Then we can build up an R-free resolution $(\mathbf{F}., \delta.)$ of M and a special filtration $\mathbb{F}$ on it such that $gr_{\mathbb{F}}(\mathbf{F}.) = \mathbf{G}.$.*

We recall that $(\mathbf{F}., \delta.)$ is defined by an inductive process. Let us present the inductive steps because the construction will be useful in the following (for more detail we refer to [39]). Starting from $(\mathbf{G}., d.)$, denote by $\{\epsilon_{0i}\}$ a basis of G_0. We put $g_i = d_0(\epsilon_{0i}) \in gr_{\mathbb{M}}(M)$ and let $f_i \in M$ be such that $gr_{\mathbb{M}}(f_i) = g_i$. Then $a_{0i} = \nu_{\mathbb{M}}(f_i)$. We define the R-free module F_0 of rank β_0 with the induced special filtration $\mathbb{F}_0$ on R^{β_0}

$$F_0 = \oplus_{i=1}^{\beta_0} R(-a_{0i})$$

Denote by $\{e_{0i}\}$ a basis of F_0 and define $\delta_0 : F_0 \to M$ such that $\delta_0(e_{0i}) = f_i$. Since d_0 is surjective, the f_i's generate a standard basis of M, $Ker(d_0) = gr_{\mathbb{F}_0}(Ker(\delta_0))$ and $F_0 \overset{\delta_0}{\to} M \to 0$ is exact. We can repeat the same procedure on the successive j-steps ($j > 0$) of the resolution of $gr_{\mathbb{M}}(M)$.

It is worth to remark that if we start from a minimal free resolution of $gr_{\mathbb{M}}(M)$, then the built up $R-$free resolution of M, is not necessary minimal. It is minimal if and only if $\beta_i(M) = \beta_i(gr_{\mathbb{M}}(M))$.

Let N be a finitely generated graded module over the polynomial ring P. We consider

$$G_j = \oplus_{i=1}^{\beta_j} P(-a_{ji}) \overset{d_j}{\to} G_{j-1} = \oplus_{i=1}^{\beta_{j-1}} P(-a_{j-1,i}),$$

a part of a *minimal* free resolution $(\mathbf{G}., d.)$ of N with $a_{j1} \leq \cdots \leq a_{j\beta_j}$ and $a_{j-1,1} \leq \cdots \leq a_{j-1,\beta_{j-1}}$. Let $1 \leq s \leq \beta_j$ and $1 \leq r \leq \beta_{j-1}$ and set

$$u_{rs} := a_{js} - a_{j-1,r},$$

then the matrix $U_j = (u_{rs})$ is called the *j-th degree-matrix* of N. We say that U_j is *non-negative* if all the entries of U_j are non-negative. We remark that the matrices U_j are univocally determined by the graded P-module N. Denote by $\mathrm{pd}(N)$ the projective dimension of N as a P-module.

Proposition 4.6. *With the above notations, let $(\mathbf{F}., \delta.)$ be a free resolution of M coming from a graded minimal free resolution $(\mathbf{G}., d.)$ of $gr_{\mathbb{M}}(M)$. If the degree-matrices U_j of $gr_{\mathbb{M}}(M)$ are non-negative for every $j \leq \mathrm{pd}(gr_{\mathbb{M}}(M))$, then $(\mathbf{F}., \delta.)$ is minimal.*

The converse of Proposition 4.6 is not true. For example consider the local ring $A = k[[t^9, t^{17}, t^{19}, t^{39}]]$ with the $\mathfrak{m}$-adic filtration where $\mathfrak{m}$ is the maximal ideal of A. One can check that the degree matrices of $\mathrm{gr}_{\mathfrak{m}}(A)$ have negative entries but the resolution of A coming from the minimal free resolution of $\mathrm{gr}_{\mathfrak{m}}(A)$ is minimal.

Example 4.7. Consider $A = k[[t^{19}, t^{26}, t^{34}, t^{40}]] \simeq R/I$ where $I \subseteq R = k[[x, y, z]]$. As we have already seen in Example 4.2, in this case $\mu(I) = \mu(I^*)$. Since both A and $gr_{\mathfrak{m}}(A)$ are Cohen-Macaulay, they have the same homological dimension, nevertheless the minimal resolutions have different Betti numbers. The corresponding minimal free resolutions have the following numerical invariants:

$$0 \to R \to R^5 \to R^5 \to R \to A \to 0$$

$$0 \to P(-5) \oplus P(-8) \to P^3(-4) \oplus P^2(-5) \oplus P(-6) \to P^5(-3) \to P \to gr_{\mathfrak{m}}(A) \to 0$$

This means that if we build the R-free resolution of A starting from a P-free minimal resolution of $gr_{\mathfrak{m}}(A)$, it is not minimal. Accordingly with Proposition 4.6 we notice that the last homogeneous map in the resolution of $gr_{\mathfrak{m}}(A)$, has degree matrix with an element of negative degree $5 - 6 = -1$.

Example 4.8. Let I be an ideal of the regular ring $(R, \mathfrak{n})$ such that $A = R/I$ is C-M. Assume

$$HS_A(z) = \frac{1 + h_1 z + h_2 z^2}{(1 - z)^d}$$

where $h_1, h_2 \in \mathbf{Z}$ and $d = \dim A$. Then A is of homogeneous type.

J. Elias and G. Valla proved that in this case $gr_{\mathfrak{m}}(A) \simeq P/I^*$ (where $\mathfrak{m} = \mathfrak{n}/I$) is C-M (see [11]). Hence we can compute the Betti numbers of $gr_{\mathfrak{m}}(A)$ from those of the Artinian reduction which has Hilbert series $1 + h_1 z + h_2 z^2$. It is easy to see that the degree matrices ($h_2 \neq 0$) are non negative. In fact

$$U_j = \left(\begin{array}{ccc|ccc} 1 & \cdots & 1 & 2 & \cdots & 2 \\ 1 & \cdots & 1 & 2 & \cdots & 2 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{array} \right)$$

Hence if we build up the free resolution $(\mathbf{F}., \delta.)$ of I all the entries of $\mathcal{M}_j$ belong to $\mathfrak{n}$.

5 Consecutive cancellations in Betti numbers of local rings

If I is a homogeneous ideal in a polynomial ring P over a field, by Macaulay's Theorem, there exists a lexicographic ideal $L = Lex(I)$ with the same Hilbert function as I. A result of Bigatti, Hulett and Pardue says that the graded Betti numbers $\beta_{ij}(P/L)$ are greater than or equal to the corresponding graded Betti numbers $\beta_{ij}(P/I)$. Peeva (see [29]) proved that the graded Betti numbers $\beta_{ij}(P/I)$ can be obtained from the graded Betti numbers $\beta_{ij}(P/L)$ by a sequence of *zero consecutive cancellations*, that is cancellations in the graded Betti numbers of consecutive homological degrees corresponding to the same shift.

As application of Section 4, the aim of this section we shall present an extension of Peeva's result to the local case.
Because the Hilbert function of the local ring $A = R/I$ is the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(A) := \oplus_{t\geq 0}\mathfrak{m}^t/\mathfrak{m}^{t+1}$ where $\mathfrak{m} = \mathfrak{n}/I$, to any ideal I corresponds the unique lexicographic ideal $L = Lex(I)$ such that P/L has the same Hilbert function as $gr_{\mathfrak{m}}(A)$. It will be enough to enlarge the allowed cancellations on the resolution of $L = Lex(I)$ for getting a resolution of a local ring R/I. Because we can pass from the resolution of L to the resolution of $gr_{\mathfrak{m}}(A) = P/I^*$ by using Peeva's result, the crucial point will be to describe the second step: the possible cancellations from a resolution of $gr_{\mathfrak{m}}(A)$ to a resolution of A.

This connection between the graded perspective and the local one is a new viewpoint and we hope it will be useful for studying the numerical invariants of local rings. We will see interesting applications.

Most of the results are given in the more general setting of the filtrations on a module M over a regular local ring $(R, \mathfrak{n})$. The results that I shall present here are containing in the paper [40].

We present the following definition which is a suitable adaptation of Peeva's definition.

Given a sequence of numbers $\{c_i\}$ such that $c_i = \sum_{j\in\mathbf{N}} c_{ij}$, we obtain a new sequence by a *consecutive cancellation* as follows: fix an index i, and choose j and j' such that $j \leq j'$ and $c_{ij}, c_{i-1,j'} > 0$; then replace c_{ij} by $c_{ij} - 1$ and $c_{i-1,j'}$ by $c_{i-1,j'} - 1$, and accordingly, replace in the sequence c_i by $c_i - 1$ and c_{i-1} by $c_{i-1} - 1$. If $j < j'$ we call it an *i negative consecutive cancellation* and if $j = j'$ an *i zero consecutive cancellation.*

A *sequence of consecutive cancellations* will mean a finite number of consecutive cancellations performed on a given sequence.

Let N be a homogeneous P-module with P- free graded resolution given by:

$$\mathbf{G}.: \ 0 \to \oplus_{i\geq 0} P^{\beta_{lj}}(-j) \stackrel{d_l}{\to} \oplus_{i\geq 0} P^{\beta_{l-1,j}}(-j) \stackrel{d_{l-1}}{\to} \dots \stackrel{d_1}{\to} \oplus_{i\geq 0} P^{\beta_{0j}}(-j).$$

According to the above definition, we will say that the sequence of the Betti numbers $\{\beta_i = \sum_{j\in\mathbf{N}} \beta_{ij}\}$ of N admits an *i negative consecutive cancellation* (resp. *i zero consecutive cancellation)* if there exist integers $j < j'$ (resp. $j = j'$) such that $\beta_{ij}, \beta_{i-1,j'} > 0$.

For example $\dots \to P(-3) \oplus P(-5) \oplus P(-6) \to P^2(-2) \oplus P^2(-5) \to \dots$ admits a zero cancellation $(P(-5), P(-5))$ and a negative cancellation $(P(-3), P(-5))$.

Notice that an i negative consecutive cancellation in a graded free resolution of N corresponds to a negative entry i-th degree-matrix of N.

Combining Theorem 4.5 and Proposition 4.6, we present the following result.

Theorem 5.1. *Let $(R, \mathfrak{n})$ be a regular local ring and let M be a filtered R-module. Then the Betti numbers of M as R-module can be obtained from the sequence of the Betti numbers of $gr_{\mathbb{M}}(M)$ as P-module by a sequence of negative consecutive cancellations.*

The crucial point of the above result is Proposition 4.6. We will sketch here the proof.

Let $(\mathbf{G}., d.)$ be the minimal free resolution of $gr_{\mathbb{M}}(M)$ and $\{\beta_i = \sum_{j\in\mathbf{N}} \beta_{ij}\}$ the corresponding sequence of the Betti numbers. By Theorem 4.5, we build up a free resolution $(\mathbf{F}., \delta.)$ of M as

R-module from $(\mathbf{G}.,d.)$. If $(\mathbf{F}.,\delta.)$ is not minimal, then, for some integer i, the matrix of the i-th differential map δ_i

$$F_i = R^{\beta_i} \xrightarrow{\delta_i} F_{i-1} = R^{\beta_{i-1}}$$

has an invertible entry. By Proposition 4.6, the i-th degree-matrix $U_i = (u_{rs})$ of $gr_{\mathbb{M}}(M)$ has a negative entry. This means that the sequence $\{\beta_i = \sum_{j\in\mathbf{N}} \beta_{ij}\}$ admits an i negative consecutive cancellation.

Then we apply to the resolution $(\mathbf{F}.,\delta.)$ a standard procedure (see for example [8]). After a suitable change of the basis of F_{i-1}, the matrices of differential maps in the resolution of M change just for δ_i and δ_{i-1}. Actually we may define a trivial subcomplex of $(\mathbf{F}.,\delta.)$

$$\mathbf{H}.: \underbrace{0 \to \cdots \to 0}_{l-i+1} \to R \xrightarrow{id} R \to \underbrace{0 \to \cdots \to 0}_{i},$$

where l is the length of $(\mathbf{F}.,\delta.)$, embedded in $\mathbf{F}.$ in such a way that $\widetilde{\mathbf{F}.} = \mathbf{F}./\mathbf{H}.$ is again a free resolution of M which corresponds to cancel a copy of R in F_i and F_{i-1}. It is easy to see that the eventually remaining invertible entries of the matrices of differential maps of new resolution still correspond to negative entries of degree matrices of $gr_{\mathbb{M}}(M)$ out of the row and column of the previous entry of negative degree. We can repeat the procedure on $\widetilde{\mathbf{F}.}$ until to reach a minimal free resolution of M.

We present the following examples in order to help the reader to visualize better the procedure.

Example 5.2. We consider again Example 4.7, i.e. $A = k[[t^{19}, t^{26}, t^{34}, t^{40}]]$. Starting from the minimal P-free resolution of $gr_{\mathfrak{m}}(A)$:

$$0 \to P(-5) \oplus P(-8) \to P^3(-4) \oplus P^2(-5) \oplus P(-6) \to P^5(-3) \to P \to gr_{\mathfrak{m}}(A) \to 0$$

we get the resolution

$$0 \to R \to R^5 \to R^5 \to R \to A \to 0$$

of A by deleting $P(-5)$ and $P(-6)$.

Example 5.3. Let $I = (x^2 + xy^3, xy + z^3, xz^3 - xy^4 + y^2z^4)$ be in $R = K[[x,y,z]]$ and consider $A = R/I$ with the maximal ideal $\mathfrak{m} = \mathfrak{n}/I$. Then $\mathrm{gr}_{\mathfrak{m}}(A) = P/I^*$ where $P = k[x,y,z]$ and I^* is the ideal generated by the initial forms of the elements of I. An easy computation shows that $I^* = (x^2, xy, xz^3, y^2z^4, -z^6, y^6z^3)$ and the minimal free resolution of $\mathrm{gr}_{\mathfrak{m}}(A)$ is as follows

$$\begin{aligned}\mathbf{G}.: \quad 0 \; &\to P(-6) \oplus P(-9) \oplus P(-11) \to P(-3) \oplus P^2(-5) \oplus P^2(-7) \oplus P(-8) \oplus P^2(-10) \\ &\to P^2(-2) \oplus P(-4) \oplus P^2(-6) \oplus P(-9) \to P\end{aligned}$$

Since $P(-11)$ does not admit cancellations $(11 - a_{2i} > 0)$, thus $depth(A) = depth(\mathrm{gr}_{\mathfrak{m}}(A)) = 0$. The minimal free resolution $(\mathbf{F}.,\delta.)$ of A

$$0 \to R \to R^3 \to R^3 \to R$$

is obtained from the resolution of $gr_{\mathfrak{m}}(A)$ after the following 5 negative consecutive cancellations on (F_3, F_2, F_1) corresponding to $(P(-6), P(-7), 0)$, $(P(-9), P(-10), 0)$, $(0, P(-3), P(-4))$, $(0, P(-5), P(-6))$, $(0, P(-8), P(-9))$.

It should be noted that there are many examples where the existence of possible consecutive cancellations does not imply the existence of an ideal for which those cancellations are realized.

Example 5.4. Let I be an ideal in the regular local ring $(R,\mathfrak{n})$ such that $A = R/I$ is Artinian with Hilbert function $\{(1,5,1,1,1)\}$ and $R/\mathfrak{n}$ has characteristic 0. Elias and Valla (see [11]) have proved that the number of the isomorphism classes of the Artinian local rings with this Hilbert function is 5. They have different Betti numbers because they correspond to the different values of the Cohen-Macaulay type $1 \leq \tau \leq 5$. Up to isomorphism, all of them have the same associated graded ring $gr_{\mathfrak{m}}(A) = P/I^*$ where

$$I^* = (x_1^5, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_3^2, x_3x_4, x_3x_5, x_4^2, x_4x_5, x_5^2)$$

in $P = k[x_1, \dots, x_5]$. Hence the minimal free resolution of $gr_{\mathfrak{m}}(A)$ is

$$\mathbf{G}. : 0 \to P^4(-6) \oplus P(-9) \to P^{20}(-5) \oplus P^4(-8) \to P^{39}(-4) \oplus P^6(-7) \to P^{36}(-3) \oplus P^4(-6) \to P^{14}(-2) \oplus P(-5) \to P.$$

By Theorem 5.1 and by Elias and Valla's result, we know that only 5 diagrams of negative consecutive cancellations can be realized, but the resolution of $gr_{\mathfrak{m}}(A)$ admits a larger number of sequences of negative consecutive cancellations.

Next example shows that we may take advantage of the generality of Theorem 5.1 by detecting the problem by using the more advantageous filtration.

Example 5.5. Consider the ideal $I = (x^2y^5, xyz^6 - z^9, y^5z^6)$ in $R = K[[x, y, z]]$. We have $gr_{\mathfrak{m}}(A) = P/I^*$ where $I^* = (x^2y^5, xyz^6, y^5z^6, y^4z^9, y^3z^{12}, y^2z^{15}, yz^{18}, z^{21})$ in $P = K[x, y, z]$. The minimal graded free resolution of I^* as P-module is

$$\mathbf{G}. : 0 \to P(-15) \oplus P(-17) \oplus P(-19) \oplus P(-21) \to P(-12) \oplus P(-13) \oplus P^2(-14) \oplus P^2(-16) \oplus P^2(-18) \oplus P^2(-20) \oplus P(-22) \to P(-7) \oplus P(-8) \oplus P(-11) \oplus P(-13) \oplus P(-15) \oplus P(-17) \oplus P(-19) \oplus P(-21)$$

We can prove that

$$gr_{\mathfrak{n}}(I) = P \oplus P/(x^2y^5) \oplus P/(x^2, xy, xz^3, z^6)$$

where $\mathfrak{n} = (x, y, z)$. Hence the minimal free resolution of $gr_{\mathfrak{n}}(I)$ is

$$\mathbf{G}. : 0 \to P(-6) \xrightarrow{d_3} P(-3) \oplus P^2(-5) \oplus P(-7) \xrightarrow{d_2} P^2(-2) \oplus P(-4) \oplus P(-6) \oplus P(-7) \xrightarrow{d_1} P^3$$

which is easier to handle. By using the procedure of Theorem 5.1, in particular after performing the negative consecutive cancelations on (F_3, F_2, F_1) corresponding to $(P(-6), P(-7), 0)$, $(0, P(-5), P(-7))$, $(0, P(-5), P(-6))$, $(0, P(-3), P(-4))$ we get the minimal free resolution of I:

$$0 \to R^2 \to R^3 \to I \to 0.$$

Combining Peeva's result with Theorem 5.1, we immediately get the following theorem.

Theorem 5.6. *Let I be an ideal of the local regular ring $(R, \mathfrak{n})$. The Betti numbers of R/I can be obtained from the Betti numbers of $P/Lex(I)$ by a sequence of negative and zero consecutive cancellations.*

We give now some possible applications.

In the following $\mu(\ )$ will denote the minimal number of generators. Next corollary extends to the local case a recent result by Hibi and Murai in [21].

Corollary 5.7. *Let $I \subseteq \mathfrak{n}^2$ be a non-zero ideal of the local regular ring $(R, \mathfrak{n})$ of dimension n. Assume $\mu(\mathrm{Lex}(I)) \leq n$, then*

1. $\dim(R/I) = n - 1$.

2. $depth(R/I) = depth(P/I^*) = depth(P/\text{Lex}(I)) = n - \mu(\text{Lex}(I))$.

3. $\mu(\text{Lex}(I)) = pd(R/I)$.

4. $\beta_h(R/I) = \beta_h(P/I^*) = \beta_h(P/\text{Lex}(I)) = 1$ *where* $h = pd(R/I)$.

As we have announced in Section 1., we get partial information on the possible Hilbert function of an Artinian Gorenstein local ring.

Corollary 5.8. *Let* $\{(1, n, h_2, ..., h_t, 1, ..., 1, 0, 0, \dots)\}$ *be the h-vector of an Artinian Gorenstein local ring* $A = R/I$*. Then* $h_t \leq n$*.*

We give evidence of the above result by the following example.

Example 5.9. We show that $\{(1, 3, 4, 4, 1, 1, 1)\}$ cannot be the $h-$vector of any Artinian Gorenstein local ring $A = K[[x, y, z]]/I$. In fact $L = Lex(I)$ should be

$$L = (x^2, xy, x^2z, xz^2, xyz, y^4, y^3z, y^2z^2, yz^3, z^7).$$

The minimal free resolution of P/L is:

$$\begin{aligned}\mathbf{G} : 0 \quad & \to P(-5) \oplus P^3(-6) \oplus P(-9) \to P(-3) \oplus P^2(-4) \oplus P^7(-5) \oplus P^2(-8) \to \\ & P^2(-2) \oplus P(-3) \oplus P^4(-4) \oplus P(-7) \to P\end{aligned}$$

However, if we consider any sequence of zero and negative consecutive cancellations, we get $\beta_3(A) \geq 2$ which contradicts the assumption that A is Gorenstein.

Another easy consequence of Theorem 5.6 is a result stated by Macaulay, later proved by Briancon and Iarrobino with different methods and very technical devices.

Corollary 5.10. *Let* $HF = \{(1, 2, ..., d, h_d, ..., h_s, 0, ..., 0)\}$ *be the Hilbert function of an Artinian Gorenstein local ring* $A = R/I$*, then for every* $j > 0$*,* $|h_j - h_{j-1}| \leq 1$*.*

The key of the above result is the following. For every $j > 0$ define

$$e_j := |h_j - h_{j-1}|$$

and let d be the initial degree of the ideal I. It is easy to see that the minimal number of generators of degree d of the corresponding lexicographic ideal L is $e_d + 1$ and, for $j > d$, the minimal number of generators of degree j of L is e_j. Notice that $\sum_{j \geq d} e_j = d$. The minimal free resolution of P/L is given by $0 \to F_2 \to F_1 \to P \to P/L \to 0$ where F_2 and F_1 have respectively rank d and $d + 1$. In particular,

$$F_2 = \oplus_{j \geq 0} P^{e_{d+j}}(-d - j - 1) \;\text{ and }\; F_1 = P^{e_d + 1}(-d) \oplus_{j \geq 1} P^{e_{d+j}}(-d - j).$$

Since $\beta_2(A) = 1$, by looking the admissible cancellations, we may easily conclude.

It should be noted that in codimension two (not necessarily Gorenstein) for all consecutive zero or negative cancellation, there exists an ideal for which those cancellations are realized. We show this by the following example.

Example 5.11. Let $HF = \{(1, 2, 3, 4, 3, 3, 3, 2, 2, 1, 0, ..., 0)\}$, then $L = \langle x^4, x^3y, x^2y^5, xy^8, y^{10} \rangle$. The minimal free resolution of P/L is

$$0 \to P(-5) \oplus P(-8) \oplus P(-10) \oplus P(-11) \to P^2(-4) \oplus P(-7) \oplus P(-9) \oplus P(-10) \to P$$

The resolution admits two negative cancellations: $(P(-5), P(-7)), (P(-8), P(-9))$ and the zero cancellation $(P(-10), P(-10))$. The sequence obtained by the zero and negative cancellations $0 \to R \to R^2 \to R$ can be realized. It is enough to modify the matrix of L by putting 1 in the positions corresponding to the cancellations. The ideal I is generated by the maximal minors of the following matrix

$$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & y^4 & 0 & 0 \\ 1 & -x & y^3 & 0 \\ 0 & 1 & -x & y^2 \\ 0 & 0 & 1 & -x \end{pmatrix}.$$

We have $I = \langle x^4 - x^2y^2 - x^2y^3 - x^2y^4 + y^6, x^3y - xy^3 - xy^4 \rangle$ and $L = \mathrm{Lex}(I)$. Furthermore, the minimal free resolution of P/I^* is obtained by performing the only zero cancellation:

$$0 \to P(-5) \oplus P(-8) \oplus P(-11) \to P^2(-4) \oplus P(-7) \oplus P(-9) \to P$$

and $I^* = \langle x^4 - x^2y^2, x^3y - xy^3, x^2y^5 - y^7, xy^8 \rangle$ is given by the maximal minors of the matrix

$$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & y^4 & 0 & 0 \\ 0 & -x & y^3 & 0 \\ 0 & 0 & -x & y^2 \\ 0 & 0 & 1 & -x \end{pmatrix}.$$

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