Maximal Hilbert Functions

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Abstract

The set of Hilbert functions of standard graded algebras is considered as a partially ordered set under numerical comparison. For the set of algebras $H(d, e_0)$, of a given dimension d and multiplicity e_0, we describe the requirements its maximal elements must satisfy; under fairly general conditions, the extremal functions arise from Cohen-Macaulay algebras. We also examine the subset $H(d, e_0, e_1)$, of those functions whose first two coefficients of their Hilbert polynomials are assigned. Finally, we show how these results and the use of certain extended multiplicities can be used to prove finiteness theorems for the number of corresponding functions.

1 Introduction

A standard graded algebra is a graded ring $G = \bigoplus_{n\geq 0} G_n$, finitely generated over G_0 by its elements of degree 1, $G = G_0[G_1]$. Unless stated otherwise, those will be the only kind we shall treat. When G_0 is an Artinian local ring, we denote the Hilbert function of G by $H_G(n) = \ell(G_n)$, where $\ell(\cdot)$ is the ordinary length function. The Hilbert–Poincaré series of G is defined as

$$P_G(t) = \sum_{n\geq 0} H_G(n)t^n.$$

This is a rational function $P_G(t) = \frac{h(t)}{(1-t)^d}$, where $h(1) = \deg(G)$, $d = \dim G$ are respectively the *degree* or *multiplicity* of G and its dimension. We note by $\mathcal{P}_G(t)$ the corresponding Hilbert polynomial

$$\mathcal{P}_G(t) = e_0(G)\binom{t+d-1}{d-1} - e_1(G)\binom{t+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}(G).$$

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In particular $e_0(G) = deg(G) = h(1)$.

There are also iterated versions of these functions and we will make use of $H^1_G(n) = \sum_{i\leq n} H_G(i)$, and the corresponding Hilbert series $P^1_G(t) = \frac{P_G(t)}{1-t}$ and Hilbert polynomial

$$\mathcal{P}^1_G(t) = e_0(G)\binom{t+d}{d} - e_1(G)\binom{t+d-1}{d-1} + \cdots + (-1)^d e_d(G)$$

The integers $e_0(G), \ldots, e_d(G)$ are called the Hilbert coefficients of G. Where there is no danger of confusion, we will write for short e_i instead of $e_i(G)$.

There is a great deal of interest on the structure of the set $\mathcal{H}$ of these functions. Our approach to them takes into account the partially ordered structure afforded by the definition

$$P_G(t) \geq P_{G'}(t) \Leftrightarrow H_G(n) \geq H_{G'}(n), \quad \forall n.$$

Thus for a given a condition $\mathfrak{C}$ on Hilbert functions, we define $H(\mathfrak{C})$ to be the set of all Hilbert functions satisfying $\mathfrak{C}$. Two of the main questions are to search for the extremal members of $H(\mathfrak{C})$ and to ascertain when it is finite. Among these sets we will consider $H(d, e_0)$, defined by all algebras with a given dimension d and multiplicity e_0, and its subset $H(d, e_0, e_1)$.

One of the most significant classes of these algebras arise as associated graded rings of filtrations of Noetherian local rings, particularly of the following kind. Let $(R, \mathfrak{m})$ be a Noetherian local and let I be an $\mathfrak{m}$–primary ideal. The Hilbert function of the associated graded ring

$$\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$$

is significant for its role as a control of the blowup process of $\mathrm{Spec}(R)$ along the subvariety $V(I)$. A challenging problem consists in relating $P_I(t) = P_{\mathrm{gr}_I(R)}(t)$ directly to R and I, as $\mathrm{gr}_I(R)$ may fail to inherit some of the arithmetical (e.g. Cohen–Macaulayness) properties of R.

We will now describe some of our results. Each deals with one of the general aspects, mentioned above, of the set of Hilbert functions of algebras of a fixed dimension. Section 2 deals with general bounds for the set $H(d, e_0(I))$ where $e_0(I) := e_0(\mathrm{gr}_I(R))$. It is centered on estimates of the following kind:

Theorem 2.2 *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}-$primary ideal in R. If $J = (x_1, \ldots, x_d)$ is an ideal generated by a system of parameters in I, then*

$$P_I(t) \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d}.$$

If the equality holds, then $\mathrm{gr}_I(R)$ is Cohen-Macaulay.

When R is Cohen–Macaulay, $\ell(R/J) = e_0(I)$, which gives the formula above a convenient expression. In this case it shows that $H(d, e_0(I), \ell(R/I))$ has a unique maximal element.

The next section considers bounds involving $e_0(I)$ and $e_1(I)$. It is generally framed by the fact that the parameters $e_0(I)$ and $e_1(I)$ are not independent of one another. In addition, for fine control, we will require Cohen–Macaulay hypotheses to sharpen the results of the previous section.

Our first general bound was partially motivated by a calculation in [3].

Theorem 3.3 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}-$primary ideal. Then*

$$P_I^1(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^{d+1}}$$

where $s = e_1(I) - e_0(I) + \ell(R/I)$.

This gets further refined when we consider the tangent cones of Cohen–Macaulay local rings. In Theorem 3.4 we prove that

$$P_R^1(t) \leq \frac{1 + bt + (e_0 - b - 2)t^2 + t^{\delta+2}}{(1-t)^{d+1}},$$

where b and δ are certain functions of e_0 and e_1.

It is quite clear how to exploit these bounds in order to derive finiteness results for the number of Hilbert functions. What is required is a mechanism to limit the postulation numbers of the algebras. To that end we employ the notion of *extended* or *cohomological* multiplicity introduced in [2]. In section 4 we give a bound on the number of Hilbert functions that have a given dimension and given extended multiplicity. This notion is however a strong requirement in comparison to the ordinary multiplicity. In counterpoint it provides effective bounds on the corresponding Hilbert coefficients:

Theorem 4.3 *Let $\mathrm{Deg}(\cdot)$ be a cohomological degree function on graded algebras. Given two positive integers A, d, there exist only a finite number of Hilbert functions associated to standard graded algebras G over Artinian rings such that $\dim G = d$ and $\mathrm{Deg}(G) \leq A$. Furthermore, there are integers b_i dependent only on $\dim G$ such that $|e_i| \leq b_i \mathrm{Deg}(G)^{i+1}$, where e_i, for $0 \leq i \leq d$, are the coefficients of the Hilbert polynomial $\mathcal{P}_G^1(t)$.*

2 Boundedness of Hilbert Functions

In this section we develop general bounds for the Hilbert functions of algebras of a given dimension d and a given multiplicity. The algebras considered throughout will be either the associated graded ring of an $\mathfrak{m}$-primary ideal of a Noetherian local ring $(R, \mathfrak{m})$, or, more generally, standard graded algebras $G = \bigoplus_{n\geq 0} G_n$ where G_0 is an Artinian local ring. For simplicity only of expression we will drop 'Noetherian'. It will be harmlessly assumed that the residue fields of these local rings are infinite. This is achieved, without changing the Hilbert functions, in the usual manner: replacing the local ring $(R, \mathfrak{m})$ by $R[X]_{\mathfrak{m}R[X]}$, where X is an indeterminate over R. For definitions and basic results we shall use [1], [5] and [15].

Let $(R, \mathfrak{m})$ be a local ring of dimension d and let I be an $\mathfrak{m}-$primary ideal in R. We denote by

$$H_I(n) = \ell(I^n/I^{n+1})$$

the Hilbert function of I. In the case $I = \mathfrak{m}$, we write $H_R(n)$. If we let

$$H_I^1(n) = \sum_{j=0}^{n} H_I(j) = \ell(R/I^{n+1})$$

then $H_I^1(n) - H_I^1(n-1) = H_I(n)$.

Let $P_I(t) = \sum_{n\geq 0} H_I(n)t^n$ be the Hilbert series of I, then

$$P_I^1(t) = \sum_{n\geq 0} H_I^1(n)t^n = \frac{P_I(t)}{(1-t)}.$$

The following proposition can be considered a consequence of a well known result proved by Singh, of which we provide a short proof.

Proposition 2.1 *Let $(R, \mathfrak{m})$ be a local ring and let I be an $\mathfrak{m}-$primary ideal in R. If $x \in I$, $\overline{R} = R/xR$ and $\overline{I} = I/(x)$, then*

1. $H_I(n) = H_{\overline{I}}^1(n) - \ell(I^{n+1} : x/I^n)$ *for every* $n \geq 0$.
2. $P_I(t) \leq \frac{P_{\overline{I}}(t)}{(1-t)}$ *and if the equality holds, then $x^* \in I/I^2$ is regular in $\operatorname{gr}_I(R)$ and $\operatorname{gr}_I(R)/(x^*) \simeq \operatorname{gr}_{\overline{I}}(\overline{R})$.*

Proof. From the exact sequence

$$0 \to (I^{n+1} : x)/I^n \longrightarrow R/I^n \longrightarrow R/I^{n+1} \longrightarrow \overline{R}/\overline{I}^{n+1} \to 0,$$

induced by multiplication by x, we get the equality 1.

For the second assertion, the first claim follows by 1. since

$$\frac{P_{\overline{I}}(t)}{(1-t)} = P_{\overline{I}}^1(t) = \sum_{n\geq 0} H_{\overline{I}}^1(n)t^n.$$

In particular, if the equality holds, then $I^{n+1} : x = I^n$ for every n, hence $x^* \in I/I^2$ is regular in $\operatorname{gr}_I(R)$. It is then well known that this implies $\operatorname{gr}_I(R)/(x^*) \simeq \operatorname{gr}_{\overline{I}}(\overline{R})$. □

Theorem 2.2 *Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}-$primary ideal in R. If $J = (x_1, \ldots, x_d)$ is an ideal generated by a system of parameters in I, then*

$$P_I(t) \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d}.$$

If the equality holds, then $\mathrm{gr}_I(R)$ is Cohen-Macaulay.

Proof. We induct on d. Let $d = 1$ and $J = (x)$ where x is a parameter in I. We have

$$\frac{\ell(R/I) + \ell(I/J)t}{(1-t)} = \ell(R/I) + \ell(R/xR)t + \ell(R/xR)t^2 + \cdots + \ell(R/xR)t^n + \cdots$$

and $H_I(0) = \ell(R/I)$.

We remark that

$$R \supseteq I^n \supseteq I^{n+1} \supseteq xI^n$$
$$R \supseteq xR \supseteq xI^n,$$

so that

$$\ell(R/xR) + \ell(xR/xI^n) = \ell(R/I^n) + H_I(n) + \ell(I^{n+1}/xI^n).$$

On the other hand, from the exact sequence

$$0 \to (0 : x + I^n)/I^n \to R/I^n \to xR/xI^n \to 0$$

we get

$$\ell(R/I^n) = \ell(xR/xI^n) + \ell((0 : x + I^n)/I^n).$$

It follows that for every $n \geq 1$

$$\ell(R/xR) = H_I(n) + \ell(I^{n+1}/xI^n) + \ell((0 : x + I^n)/I^n). \quad (1)$$

This proves that $H_I(n) \leq \ell(R/xR)$ for every $n \geq 1$ and the first assertion of the theorem follows.

If we have the equality

$$P_I(t) = \frac{\ell(R/I) + \ell(I/J)t}{(1-t)},$$

then $\ell(R/xR) = H_I(n)$ for every $n \geq 1$, so that, by (1),

$$\ell(I^{n+1}/xI^n) = \ell((0 : x + I^n)/I^n) = 0.$$

This implies that $x^* \in I/I^2$ is regular in $\mathrm{gr}_I(R)$ and $\mathrm{gr}_I(R)$ is Cohen-Macaulay.

Suppose $d \geq 2$, and let $\overline{R} = R/x_1R$, $\overline{I} = I/x_1R$, $\overline{J} = J/x_1R$. Then $\overline{R}/\overline{I} \simeq R/I$ and $\overline{I}/\overline{J} \simeq I/J$; further $\overline{I}$ is a primary ideal in the local ring $\overline{R}$ which has $\dim \overline{R} = d-1$. By induction we have

$$P_{\overline{I}}(t) \leq \frac{\ell(\overline{R}/\overline{I}) + \ell(\overline{I}/\overline{J})t}{(1-t)^{d-1}} = \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^{d-1}}.$$

Since $1/(1-t) \geq 0$, we also get

$$\frac{P_{\overline{I}}(t)}{(1-t)} \leq \frac{\ell(\overline{R}/\overline{I}) + \ell(\overline{I}/\overline{J})t}{(1-t)^{d}},$$

hence, using Proposition 2.1, we get

$$P_I(t) \leq \frac{P_{\overline{I}}(t)}{(1-t)} \leq \frac{\ell(\overline{R}/\overline{I}) + \ell(\overline{I}/\overline{J})t}{(1-t)^{d}} = \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^{d}}.$$

This proves the first assertion of the theorem for any $d \geq 1$.

If we have the equality

$$P_I(t) = \frac{\ell(R/I) + \ell(I/J)t}{(1-t)d},$$

then

$$P_I(t) = \frac{P_{\overline{I}}(t)}{(1-t)}, \quad P_{\overline{I}}(t) = \frac{\ell(\overline{R}/\overline{I}) + \ell(\overline{I}/\overline{J})t}{(1-t)^{d-1}}.$$

The first equality implies by Proposition 2.1 that $x_1^* \in I/I^2$ is regular in $\mathrm{gr}_I(R)$ and $\mathrm{gr}_I(R)/(x_1^*) \simeq \mathrm{gr}_{\overline{I}}(\overline{R})$.

The second equality implies by induction that $\mathrm{gr}_{\overline{I}}(\overline{R}) \simeq \mathrm{gr}_I(R)/(x_1^*)$ is Cohen-Macaulay. Hence $\mathrm{gr}_I(R)$ is Cohen-Macaulay. □

Remark 2.3 If the equality holds in the above theorem, then R itself is Cohen-Macaulay and $\ell(R/J) = e_0(I)$. In particular I has the minimal possible multiplicity for an $\mathfrak{m}-$primary ideal in R.

Nearly the same treatment applies to standard graded algebras, which we state for later reference.

Proposition 2.4 *Let $(G_0, \mathfrak{m})$ be an Artinian local ring and let $G = \bigoplus_{n \geq 0} G_n$ be a standard graded algebra of dimension $d \geq 1$. If J is the ideal generated by an homogeneous system of parameters in G, then*

$$P_G(t) \leq \frac{\ell(G_0) + (\ell(G/J) - \ell(G_0))t}{(1-t)^d}.$$

If the equality holds, then G is Cohen-Macaulay.

Proof. Set $I = G_+ = \bigoplus_{n\geq 1} G_n$, and note that I is primary for the irrelevant maximal ideal M of G, $\mathrm{gr}_I(G) \simeq G$, and that J is generated by a system of parameters. Note also that the associated graded rings and lengths are not changed if G or the localization G_M are considered. Theorem 2.2 can now be applied directly. □

These results show that the Hilbert function of I is bounded by the rational function

$$\frac{\ell(R/I) + (\ell(R/J) - \ell(R/I))t}{(1-t)^d}$$

for any ideal J generated by a system of parameters that yields minimal length for R/J. It is however not clear which number this turns out to be, except when R is Cohen–Macaulay when we have:

Corollary 2.5 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$. If I is an $\mathfrak{m}-$primary ideal in R, then*

$$P_I(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I))t}{(1-t)^d}.$$

If the equality holds, then $\mathrm{gr}_I(R)$ is Cohen-Macaulay.

Proof. Under our assumptions, if $x_1, \ldots, x_d$ is a superficial sequence for I, then it is a system of parameters in I and $\ell(R/J) = e_0(I)$. □

This formula, in case $I = \mathfrak{m}$, was obtained in [2] through different means. We shall now explain the difference between the two sides of the formula above. Let J be a minimal reduction of the ideal I. Since R has an infinite residue field, J is generated by a regular sequence. Now consider the construction of the Sally module of I relative to J: it is simply the $R[Jt]$–module $S_J(I)$ defined by the natural exact sequence

$$0 \to I \cdot R[Jt] \longrightarrow I \cdot R[It] \longrightarrow S_J(I) \to 0.$$

$S_J(I) = 0$ when $I^2 = JI$, that is when I has so–called *minimal multiplicity.* In all the other cases, $S_J(I)$ has no embedded primes and $\dim S_J(I) = \dim R = d$.

A calculation in [20] (see also [21]) shows that

$$P_I(t) = \frac{\ell(R/I) + (e_0(I) - \ell(R/I))t}{(1-t)^d} - (1-t)P_{S_J(I)}(t).$$

Hence the inequality of Corollary 2.5 is equivalent to the following assertion:

Corollary 2.6 *The Hilbert function of $S_J(I)$ is non–decreasing.*

This answers a question raised in [19, p. 385].

Remark 2.7 One application of Theorem 2.2 is to employ the technique of [2] to obtain estimates for the reduction number r of the ideal I.

For simplicity of notation, set $a := \ell(R/I)$, $b := \ell(I/J)$ and $c := a + b = \ell(R/J)$. From the inequality of Hilbert series

$$P_I(t) \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d},$$

we have that for each positive integer n,

$$\ell(I^n/\mathfrak{m}I^n) \leq \ell(I^n/I^{n+1}) \leq a\binom{n+d-1}{d-1} + b\binom{n+d-2}{d-1}.$$

According to [4], if for some integer n we bound the right hand side of this inequality by $\binom{n+d}{d}$, we can find a reduction J of I such that $JI^{n-1} = I^n$. This is easy to work out since the inequality

$$a\binom{n+d-1}{d-1} + b\binom{n+d-2}{d-1} < \binom{n+d}{d}$$

is quadratic:

$$(n+d)(n+d-1) > ad(n+d-1) + bdn.$$

The inequality is certainly satisfied for $n > dc - 2d + 1 + \sqrt{(a-1)(d-1)d}$. As a consequence we get a bound for the reduction number r of I, namely

$$r \leq dc - 2d + 1 + \sqrt{(a-1)(d-1)d}.$$

The inequality $r \leq dc - 2d + 1$ is the bound in [2] for the Cohen–Macaulay case, so that $\sqrt{(a-1)(d-1)d}$ is a penalty for the lack of that condition.

3 Bounds involving $e_0(I)$ and $e_1(I)$

Throughout this section $(R, \mathfrak{m})$ will be a Cohen–Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}-$primary ideal and we denote by $e_0(I)$ and $e_1(I)$ the first two coefficients of the Hilbert polynomial of I.

The integers $e_0(I)$ and $e_1(I)$ are loosely related. In case $I = \mathfrak{m}$ (see [10]):

$$e_0(\mathfrak{m}) - 1 \leq e_1(\mathfrak{m}) \leq \binom{e_0(\mathfrak{m}) - 1}{2}.$$

Actually there are more strict relations when Hilbert functions of primary ideals are considered. The following proposition gives an instance.

We need to recall some basic properties of one dimensional Cohen-Macaulay local rings. Let R be a Cohen-Macaulay local ring of dimension one. If x is a superficial element in I, then for every non negative integer j

$$H_I(j) = e_0(I) - v_j$$

where $v_j = \ell(I^{j+1}/xI^j)$ (see (1)). In particular $v_0 = e_0(I) - \ell(R/I)$, $v_1 = e_0(I) - \ell(I/I^2)$ and if $v_n = 0$ for some n, then $v_j = 0$ for every $j \geq n$. It is also known (see [15, Theorem 6.18]) that

$$e_1(I) = \sum_{j \geq 0} v_j.$$

Proposition 3.1 *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring of dimension one, let I be an $\mathfrak{m}$–primary ideal. If $e_0(I) \neq e_0(\mathfrak{m})$ then*

$$e_1(I) \leq \binom{e_0(I) - 2}{2}.$$

Proof. The condition $e_0(I) \neq e_0(\mathfrak{m})$ means, by the theorem of Rees (see [12]), that $\mathfrak{m}$ is not the integral closure of I. This implies that for each positive integer n, $I^{n+1} \neq \mathfrak{m}I^n$, and therefore $\ell(I^n/I^{n+1}) > \ell(I^n/\mathfrak{m}I^n)$.

If x is a superficial element of I, then x is a regular element in R so that by (1) the Hilbert function of I can be written

$$H_I(n) = \ell(I^n/I^{n+1}) = e_0(I) - v_n,$$

so it only reaches its stable value of $e_0(I)$ when $n = r$, the reduction number of I, that is the smallest r for which $I^{r+1} = xI^r$. We claim that for all $n \leq r$, $\ell(I^n/I^{n+1}) \geq n + 2$. Indeed otherwise we would have $\ell(I^n/\mathfrak{m}I^n) \leq n$, which by the main theorem of [4] would lead to an equality $I^n = xI^{n-1}$, contradicting the definition of r. This means that we have

$$\begin{aligned} e_1(I) &= \sum_{n=0}^{r} (e_0(I) - \ell(I^n/I^{n+1})) \\ &\leq e_0(I) - \ell(R/I) + \sum_{n=1}^{r} (e_0(I) - (n+2)) \\ &= e_0(I) - \ell(R/I) + r(e_0(I) - 2) - \binom{r+1}{2}. \end{aligned}$$

Since $r \leq e_0(I) - 1$, we may assume that $e_0(I) \geq 3$ (otherwise $I = (x)$ and the result follows). Substituting we have the desired inequality. □

The following result on the Hilbert series of a Cohen-Macaulay local ring of dimension one will be crucial in different applications.

We define

$$\delta := e_1(I) - 2e_0(I) + \ell(R/I) + \ell(I/I^2)$$

and we remark that

$$\delta = e_1(I) - v_0 - v_1 = \sum_{j\geq 2} v_j \geq 0.$$

Proposition 3.2 *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension one and let I be an $\mathfrak{m}-$primary ideal. We set $h = \ell(I/I^2) - \ell(R/I)$ and let δ be the integer defined above. Then*

$$P^1_I(t) \leq \frac{\ell(R/I) + ht + (e_0(I) - h - \ell(R/I) - 1)t^2 + t^{\delta+2}}{(1-t)^2}.$$

Proof. We have

$$\frac{\ell(R/I) + ht + (e_0(I) - h - \ell(R/I) - 1)t^2 + t^{\delta+2}}{(1-t)^2} =$$

$$= \ell(R/I) + \sum_{n\geq 1}\Big[(n-1)e_0(I) + h + 2\ell(R/I) - \min\{n-1,\delta\}\Big]t^n.$$

Since $H^1_I(0) = \ell(R/I)$ and

$$H^1_I(1) = H_I(0) + H_I(1) = \ell(R/I) + \ell(R/I) + h = 2\ell(R/I) + h,$$

we may assume $n \geq 2$. Then we can write

$$H^1_I(n) = (n+1)e_0(I) - \sum_{j=0}^{n} v_j = (n-1)e_0(I) + h + 2\ell(R/I) - \sum_{j=2}^{n} v_j$$

and we must prove

$$\sum_{j=2}^{n} v_j \geq \min\{n-1,\delta\}.$$

The inequality follows since if $v_n \neq 0$, then $\sum_{j=2}^{n} v_j \geq n-1$, otherwise $\sum_{j=2}^{n} v_j = \sum_{j\geq 2} v_j = \delta$. □

Unless $I = \mathfrak{m}$, the above Proposition does not extend to the higher dimensional case because h and δ have a bad behavior modulo a superficial element. Hence, in the following theorem, the main point is to work out from the above proposition an upper bound for $P^1_I(t)$ which, even weaker, does not involve any longer the integers h and δ.

We recall that if I is primary for the maximal ideal of the Cohen-Macaulay local ring R, then

$$e_1(I) \geq e_0(I) - \ell(R/I).$$

Theorem 3.3 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$, let I be an $\mathfrak{m}$–primary ideal and $s := e_1(I) - e_0(I) + \ell(R/I)$.*

a) If $d = 1$, then

$$P_I^1(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^2}.$$

b) If $d \geq 2$ then

$$P_I(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^d}.$$

c) If the equality holds in a) or in b), then either

$$s = 0, \quad e_0(I) = \ell(I/I^2) + (1-d)\ell(R/I)$$

and $\mathrm{gr}_I(R)$ *is Cohen-Macaulay, or*

$$s \geq 1, \quad e_0(I) = \ell(I/I^2) + (1-d)\ell(R/I) + 1$$

and depth $\mathrm{gr}_I(R) \geq d - 1$.

Proof. We first prove a). Let $d = 1$, then we have

$$\frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^2} = \ell(R/I) + \sum_{n \geq 1} \Big[n e_0(I) + \ell(R/I) - \min\{n, s\} \Big] t^n.$$

With δ and v_i as above, we have $s = e_1(I) - e_0(I) + \ell(R/I) = \sum_{j \geq 1} v_j$ and $\delta = \sum_{j \geq 2} v_j$. By Proposition 3.2 it is enough to show that for every $n \geq 1$

$$(n-1)e_0(I) + \ell(I/I^2) + \ell(R/I) - \min\Big\{n-1, \sum_{j \geq 2} v_j\Big\} \leq n e_0(I) + \ell(R/I) - \min\Big\{n, \sum_{j \geq 1} v_j\Big\}$$

or equivalently

$$e_0(I) \geq \ \ell(I/I^2) + \min\Big\{n, \sum_{j \geq 1} v_j\Big\} - \min\{n-1, \sum_{j \geq 2} v_j\}.$$

Now $e_0(I) = H_I(1) + v_1 = \ell(I/I^2) + v_1$ and the conclusion follows since clearly

$$v_1 + \min\Big\{n-1, \sum_{j \geq 2} v_j\Big\} \geq \min\Big\{n, \sum_{j \geq 1} v_j\Big\}.$$

¿From this computation it follows that

$$P_I^1(t) = \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^2}$$

if and only if

$$v_1 + \min\Big\{n-1, \sum_{j\geq 2} v_j\Big\} = \min\Big\{n, \sum_{j\geq 1} v_j\Big\}.$$

This can happen if and only if $v_1 \leq 1$. If $v_1 = 0$, then $I^2 = xI$, hence $\mathrm{gr}_I(R)$ is Cohen-Macaulay, $e_0(I) = \ell(I/I^2)$ and $s = 0$. If $v_1 = 1$, then $s \geq 1$ and $e_0 = \ell(I/I^2) + 1$.

This proves a) and the case $d = 1$ in c).

We prove now b) by induction on $d \geq 2$. If x is a superficial element for I, then $\overline{R} = R/(x)$ is a local Cohen–Macaulay ring of dimension $d-1$. In particular $\overline{I} = I/(x)$ is primary for the maximal ideal of $\overline{R}$ and

$$e_0(I) = e_0(\overline{I}), \quad e_1(I) = e_1(\overline{I}), \quad \ell(R/I) = \ell(\overline{R}/\overline{I}).$$

This implies

$$s = e_1(I) - e_0(I) + \ell(R/I) = e_1(\overline{I}) - e_0(\overline{I}) + \ell(\overline{R}/\overline{I})$$

and

$$\frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^d} = \frac{\ell(\overline{R}/\overline{I}) + (e_0(\overline{I}) - \ell(\overline{R}/\overline{I}) - 1)t + t^{s+1}}{(1-t)^d}.$$

By Proposition 2.1, we have $P_I(t) \leq \frac{P_{\overline{I}}(t)}{1-t}$, and we claim that

$$\frac{P_{\overline{I}}(t)}{1-t} \leq \frac{\ell(\overline{R}/\overline{I}) + (e_0(\overline{I}) - \ell(\overline{R}/\overline{I}) - 1)t + t^{s+1}}{(1-t)^d}.$$

This would imply

$$P_I(t) \leq \frac{P_{\overline{I}}(t)}{1-t} \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^d} \tag{2}$$

which gives the conclusion.

To prove the claim we remark that $\frac{P_{\overline{I}}(t)}{1-t} = P_{\overline{I}}^1(t)$, so that the inequality follows by a) if $d = 2$. If $d \geq 3$, the inequality follows as well since, by induction,

$$P_{\overline{I}}(t) \leq \frac{\ell(\overline{R}/\overline{I}) + (e_0(\overline{I}) - \ell(\overline{R}/\overline{I}) - 1)t + t^{s+1}}{(1-t)^{d-1}}.$$

Let us prove now c) by induction on $d \geq 2$. First we show that the assumption implies depth $gr_I(R) \geq d-1$. If we have

$$P_I(t) = \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^d},$$

the inequalities in (2) are in fact equalities so that

$$P_I(t) = \frac{P_{\overline{I}}(t)}{1-t}, \qquad P_{\overline{I}}(t) = \frac{\ell(\overline{R}/\overline{I}) + (e_0(\overline{I}) - \ell(\overline{R}/\overline{I}) - 1)t + t^{s+1}}{(1-t)^{d-1}}.$$

The first equality implies by Proposition 2.1 that x_1^* is a regular element in $\mathrm{gr}_I(R)$ so that the conclusion follows if $d = 2$.

If $d \geq 3$, the second equalities implies by induction that depth $\mathrm{gr}_{\overline{I}}(\overline{R}) \geq d-2$, hence depth $(\mathrm{gr}_I(R)/(x_1^*)) \geq d-2$. Since x_1^* is a regular element in $\mathrm{gr}_I(R)$, we get depth $\mathrm{gr}_I(R) \geq d-1$, as desired.

We can now complete the proof of c). Let $J = (x_1, \ldots, x_{d-1})$ be a superficial sequence in I. Since depth $\mathrm{gr}_I(R) \geq d-1$, the linear forms $x_1^*, \ldots, x_{d-1}^*$ are a regular sequence in $\mathrm{gr}_I(R)$. This implies $\mathrm{gr}_{I/J}(R/J) \simeq gr_I(R)/(x_1^*, \ldots, x_{d-1}^*)$ and $P_I(t) = P_{I/J}(t)/(1-t)^{d-1}$, from which we get

$$P_{I/J}(t) = \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)}.$$

We have $\dim(R/J) = 1$ and $e_0(I) = e_0(I/J)$, $e_1(I) = e_1(I/J)$, $\ell(R/I) = \ell\left((R/J)/(I/J)\right)$, so that $s(I) = s(I/J)$. Hence we have two possibilities: either

a) $s = 0$, $e_0(I/J) = \ell\left((I/J)/(I/J)^2\right)$ and $\mathrm{gr}_{I/J}(R/J)$ is Cohen-Macaulay,

or

b) $s \geq 1$ and $e_0(I/J) = \ell\left((I/J)/(I/J)^2\right) + 1$.

The conclusion follows since $x_1, \ldots, x_{d-1}$ is a regular sequence in R so that we have $J/IJ \simeq (R/I)^{d-1}$ which implies

$$\ell\left((I/J)/(I/J)^2\right) + (d-1)\ell(R/I) = \ell(I/I^2).$$

□

We may improve the previous bound in case of the maximal ideal. Let R be a local Cohen–Macaulay ring of dimension d which is not regular; we will write $H_R(n)$ and $P_R(t)$ instead of $H_{\mathfrak{m}}(n)$ and $P_{\mathfrak{m}}(t)$. If h is the embedding codimension of R, that is $h = H_R(1) - d$, the following inequalities hold (see [8] and [6])

$$2e_0(\mathfrak{m}) - h - 2 \leq e_1(\mathfrak{m}) \leq \binom{e_0(\mathfrak{m})}{2} - \binom{h}{2}$$

We define

$$\begin{aligned} b &= \max\{n : \binom{n}{2} \leq \binom{e_0(\mathfrak{m})}{2} - e_1(\mathfrak{m})\} \\ \alpha &:= e_1(\mathfrak{m}) - 2e_0(\mathfrak{m}) + b + 2. \end{aligned}$$

Since $\binom{h}{2} \leq \binom{e_0(\mathfrak{m})}{2} - e_1(\mathfrak{m})$, we have $b \geq h$. In particular $\alpha = e_1(\mathfrak{m}) - 2e_0(\mathfrak{m}) + b + 2 \geq e_1(\mathfrak{m}) - 2e_0(\mathfrak{m}) + h + 2 \geq 0$.

Theorem 3.4 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$. We denote by b and α the integers defined above. Then*

a) If $d = 1$

$$P_R^1(t) \leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^2}.$$

b) If $d \geq 2$, then

$$P_R(t) \leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^d}.$$

If the equality holds, then depth $\mathrm{gr}_{\mathfrak{m}}(R) \geq d - 1$.

Proof. We prove a). One has

$$\frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^2} = 1 + \sum_{n \geq 1} [(n-1)e_0(\mathfrak{m}) + b + 2 - \min\{n-1, \alpha\}]t^n.$$

If we apply Proposition 3.2 in the case $I = \mathfrak{m}$ we obtain

$$\begin{aligned} P_R^1(t) &\leq \frac{1 + ht + (e_0(\mathfrak{m}) - h - 2)t^2 + t^{\delta+2}}{(1-t)^2} \\ &= 1 + \sum_{n \geq 1} [(n-1)e_0(\mathfrak{m}) + h + 2 - \min\{n-1, \delta\}]t^n, \end{aligned}$$

where $\delta = e_1(\mathfrak{m}) - 2e_0(\mathfrak{m}) + h + 2$. We have to prove that for every $n \geq 1$

$$h - \min\{n-1, \delta\} \leq b - \min\{n-1, \alpha\}.$$

Since $\alpha = b - h + \delta$ and $b \geq h$, we have $\alpha \geq \delta$. The assertion easily follows.

Part b) can be proved exactly in the same way as the corresponding statement was proved in Theorem 3.3. $\square$

Remark 3.5 If we apply Corollary 2.5 with $I = \mathfrak{m}$, we obtain

$$P_R(t) \leq \frac{1 + (e_0(\mathfrak{m}) - 1)t}{(1-t)^d}.$$

We remark that, if $d \geq 2$, the above theorem improves this bound. To prove this, note that $\frac{1}{(1-t)^{d-2}} \geq 0$, so that we only need to prove that

$$\frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^2} \leq \frac{1 + (e_0(\mathfrak{m}) - 1)t}{(1-t)^2}$$

that is

$$(n-1)e_0(\mathfrak{m}) + b + 2 - \min\{n-1, \alpha\} \leq ne_0(\mathfrak{m}) + 1$$

or equivalently

$$e_0(\mathfrak{m}) - b - 1 + \min\{n-1, \alpha\} \geq 0$$

Since $\binom{b}{2} \leq \binom{e_0(\mathfrak{m})}{2} - e_1(\mathfrak{m})$, we have $e_0(\mathfrak{m}) \geq b + 1$ and the assertion follows.

Remark 3.6 Part b) both in Theorem 3.3 and 3.4 does not hold if $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring of dimension one.

Let us consider the Cohen-Macaulay local ring $R = k[[u^5, u^6, u^7]]$. It is easy to see that $P_R(t) = (1 + 2t + 2t^2)/(1-t)$, $e_0(\mathfrak{m}) = 5$ and $e_1(\mathfrak{m}) = 6$, so that $b = 3$ and $\alpha = 1$ in the statement of Theorem 3.4. On the other hand if we take $I = \mathfrak{m}$ in Theorem 3.3, we get $s = 2$. It is clear that

$$\frac{1 + 2t + 2t^2}{1-t} \not\leq \frac{1 + 3t + t^3}{1-t},$$

but

$$P_R^1(t) = \frac{1 + 2t + 2t^2}{(1-t)^2} \leq \frac{1 + 3t + t^3}{(1-t)^2}.$$

With the usual notations, for every Cohen-Macaulay local ring R of dimension one, we know that $H_R(1) = h + 1 \leq b + 1$ and $H_R(n) \leq e_0(\mathfrak{m})$ for every $n \geq 2$. This gives the inequality

$$P_R(t) \leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 1)t^2}{(1-t)}.$$

¿From this we get a bound for $P_R^1(t)$ which is weaker than that proved in a) of Theorem 3.4. Namely we have

$$\frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^2} \leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 1)t^2}{(1-t)^2}. \quad (3)$$

On the other hand, we can easily extend the above inequality to the higher dimensional case, proving that for any Cohen-Macaulay local ring R of dimension d, we have

$$P_R(t) \leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 1)t^2}{(1-t)^d}.$$

However, because of (3), this bound is weaker than that proved in b) of Theorem 3.4.

Remark 3.7 It easy to see that the bounds given in Theorem 3.4 are tight. Let us consider the Cohen-Macaulay local ring $R = k[[u^3, u^4, u^5]]$. We have

$$P_R(t) = (1+2t)/(1-t), \quad e_0(\mathfrak{m}) = 3, \quad e_1(\mathfrak{m}) = 2,$$

so that $b = 2$ and $\alpha = 0$. We have

$$1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2} = 1 + 2t,$$

hence R verifies equality in a), while, if $d \geq 2$, $R[[Y_1, \ldots, Y_{d-1}]]$ verifies equality in b).

4 Finiteness of Hilbert Functions

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local of dimension d and let I be an $\mathfrak{m}$–primary ideal of multiplicity $e_0 = e_0(I)$. Denote by $H(d, e_0)$ the set of all Hilbert functions of the algebras $\mathrm{gr}_I(R)$. In [13] and [14] it is proved that $H(d, e_0)$ is a finite set. Its difficult proof is accomplished by providing very large bounds on the coefficients of the Hilbert polynomials and on the Castelnuovo–Mumford regularity of the algebra $\mathrm{gr}_I(R)$ in terms of e_0. As their authors point out, the assertion fails if R is not Cohen–Macaulay.

Our result in this section shows that using a different notion of multiplicity one obtains a weaker finiteness theorem which applies to arbitrary graded algebras.

Let S be either a graded algebra generated by its elements of degree 1 or a local ring. An *extended degree* (see [2]) is a function $\mathrm{Deg}(\cdot)$ on finitely generated S–modules (graded in the case of the former ring) satisfying the following conditions:

(i) If $L = \Gamma_{\mathfrak{m}}(M)$ is the submodule of elements of M which are annihilated by a power of the maximal ideal (maximal irrelevant ideal in the graded case) and $\overline{M} = M/L$, then

$$\mathrm{Deg}(M) = \mathrm{Deg}(\overline{M}) + \ell(L).$$

(ii) (Bertini's rule) If S has positive depth and $h \in S$ is a generic hyperplane section on M, then

$$\mathrm{Deg}(M) \geq \mathrm{Deg}(M/hM).$$

(iii) (The calibration rule) If M is a Cohen–Macaulay module, then

$$\mathrm{Deg}(M) = e_0(M),$$

where $e_0(M)$ is the ordinary multiplicity of the module M.

In [18] an instance of such functions was constructed:

Definition 4.1 Let M be a finitely generated graded module over the graded algebra A and let S be a Gorenstein graded algebra mapping onto A, with maximal graded ideal $\mathfrak{m}$. Assume that $\dim S = r$, $\dim M = d$. The *homological degree* of M is the integer

$$\mathrm{hdeg}(M) \quad = \quad e_0(M) + \sum_{i=r-d+1}^{r} \binom{d-1}{i-r+d-1} \cdot \mathrm{hdeg}(\mathrm{Ext}^i_S(M,S))$$

This expression becomes more compact when $\dim M = \dim S = d > 0$:

$$\mathrm{hdeg}(M) = e_0(M) + \sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \mathrm{hdeg}(\mathrm{Ext}^i_S(M,S)).$$

It is important to note that $\mathrm{hdeg}(\cdot)$ is defined recursively on the dimension of the module; we refer to [18] and [2] for more technical aspects of these definitions.

Given any cohomological degree, [9] proposed a method to construct another extended degree function where equality holds in the Bertini's condition (4). We now restate Proposition 2.4 in the language of these functions.

Corollary 4.2 *Let $G = \bigoplus_{n\geq 0} G_n$ be a standard graded algebra over an Artinian ring and let $\mathrm{Deg}(\cdot)$ be any extended degree function defined on G. If $\dim G = d \geq 1$ then*

$$P_G(t) \leq \frac{\ell(G_0) + (\mathrm{Deg}(G) - \ell(G_0)) \cdot t}{(1-t)^d}.$$

In other words, for all $n \geq 0$

$$H_G(n) \leq \mathrm{Deg}(G)\binom{d+n-2}{d-1} + \ell(G_0)\binom{d+n-2}{d-2}.$$

Proof. Let J be an ideal that is generated by a system of parameters of degree 1 that is generic for the function $\mathrm{Deg}(\cdot)$ chosen; according to [2, Proposition 2.3], $\ell(G/J) \leq \mathrm{Deg}(G)$. Now replace $\ell(G/J)$ by $\mathrm{Deg}(G)$ in the estimate of Proposition 2.4. □

We can prove now the main theorem of this section in which we bound the coefficients of the Hilbert polynomial of a graded standard algebra solely in terms of the extended degree. The bounds are explicit but far from being strict since we are only looking for the application to the finiteness of Hilbert functions.

As usual, we let $\mathrm{Deg}(\cdot)$ be an extended degree function on graded algebras.

Theorem 4.3 *Let G be a standard graded algebra over an Artinian ring G_0. For every $0 \leq i \leq d$ we define recursively the integers $b_0 := 1$, $b_i := i+1+\sum_{j=0}^{i-1}(i-j+1)b_j$ and we let $e_i := e_i(G)$. Then, for every $0 \leq i \leq d$, we have*

$$|e_i| \leq b_i \mathrm{Deg}(G)^{i+1}.$$

Proof. We induct on d. If $d = 0$, then $e_0(G) = \mathrm{Deg}(G)$ and $b_0 = 1$. For $d \geq 1$, we set $\overline{G} = G/hG$ where $h \in G_1$ is a generic hyperplane which is then a parameter in G. We have $\dim \overline{G} = d-1$, $e_i(G) = e_i(\overline{G})$ for $i = 0, \ldots d-1$ and $\mathrm{Deg}(\overline{G}) \leq \mathrm{Deg}(G)$.

By the induction hypothesis, for $i < d$ we have

$$|e_i| \leq b_i \mathrm{Deg}(\overline{G})^{i+1} \leq b_i \mathrm{Deg}(G)^{i+1}.$$

We recall now that the difference between the Hilbert function of G and its Hilbert polynomial is given by ([1, Theorem 4.3.5(b)])

$$H_G(n) - \mathcal{P}_G(n) = \sum_{i=0}^{d} (-1)^i \ell(H^i_{G_+}(G)_n). \tag{4}$$

We now make a key point on the vanishing of $H^i_{G_+}(G)_n$, for $n \geq 0$. If we denote by $a_i(G)$ the largest n for which this group does not vanish, we have the well-known description of the Castelnuovo–Mumford regularity of the algebra G,

$$\mathrm{reg}(G) = \sup\{a_i(G) + i \mid i \geq 0\}.$$

However, according to [2] $\mathrm{Deg}(G) > \mathrm{reg}(G)$ so that $H_G(n) = \mathcal{P}_G(n)$ for $n \geq \mathrm{Deg}(G)$.

This implies $H^1_G(n) = \mathcal{P}^1_G(n)$ for $n \geq \mathrm{Deg}(G) - 1$. If we let $r := \mathrm{Deg}(G)$ we get

$$|e_d| \leq H^1_G(r) + \left| \sum_{i=0}^{d-1} (-1)^i e_i \binom{d+r-i}{d-i} \right|,$$

where now we bound $H^1_G(r)$ by using the estimate given in Corollary 4.2,

$$H^1_G(r) \leq r\binom{d+r-1}{d} + \ell(G_0)\binom{d+r-1}{d-1}.$$

It follows that

$$|e_d| \leq r\binom{d+r-1}{d} + \ell(G_0)\binom{d+r-1}{d-1} + \sum_{i=0}^{d-1} |e_i| \binom{d+r-i}{d-i}.$$

Now, as in the proof of Corollary 4.2, we have

$$\ell(G_0) = \ell(G/G_+) \leq \ell(G/J) \leq r.$$

We will use the inequalities

$$r \leq r^2, \quad \binom{d+r-i}{d-i} \leq r^{d-i}(d-i+1), \quad \binom{d+r-1}{d} \leq r^d$$

and we get

$$|e_d| \leq r^{d+1} + r^{d+1}d + \sum_{i=0}^{d-1} b_i r^{i+1} r^{d-i}(d-i+1) = r^{d+1} b_d.$$

This gives the desired assertion. □

Corollary 4.4 *Given two positive integers A and d, there exists only a finite number of Hilbert functions associated to standard graded algebra G over Artinian rings such that* $\dim G = d$ *and* $\mathrm{Deg} G \leq A$.

Proof. The finiteness of the number of Hilbert functions follows from the finiteness of the possible Hilbert polynomials, after remarking that Corollary 4.2 takes care of the initial values of the Hilbert function. □

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